A homotopical approach to KK-theory

Sigurd Segtnan

Series of dissertations submitted to the
Faculty of Mathematics and Natural Sciences, University of Oslo
No. 1391

ISSN 1501-7710

Cover: Inger Sandved Anfinsen.
Printed in Norway: AIT Oslo AS.

Produced in co-operation with Akademika Publishing.
The thesis is produced by Akademika Publishing merely in connection with the thesis defence. Kindly direct all inquiries regarding the thesis to the copyright holder or the unit which grants the doctorate.

Acknowledgements

At the end of my period as a Ph.D. student there are several persons I ought to thank. First and foremost is my advisor professor Paul Arne Østvær whose door has always been open. I would also like to thank the research group in operator algebras at the university of Oslo. Moreover, I am grateful for the time I spent at the focused semester on KK-theory in Münster, and I would like to thank its organisers and participants.

There is a long list of master and Ph.D. students whose presence at Blindern has made this thesis possible. I will not try to give an exhaustive list, but my colleagues Karoline and Simen have to be mentioned. Furthermore, I am indebted to Robin for his comment on the internal smash product.

The administrative staff at the department of mathematics has always been helpful. My gratitude also goes to Inger Christin and Leyla who ran the mathematics library. On a related note I also wish to thank the staff at the library of science, whom it has been my pleasure to call colleagues for the last six months. Lastly, I would like to thank my friends and family for their support.

Contents

Introduction

One of the goals in the study of spaces is that of classification. Since the classification up to homeomorphism is impossible, the focus has been to classify up to other equivalences. In the cases of homotopy equivalence and weak homotopy equivalence this again reduces the classification problem to the study of the corresponding homotopy category. An important tool in this classification is model categories, which were developed by Daniel G. Quillen[Qui67]. The main point of this theory is that instead of studying the homotopy category, one can study a model for the homotopy category. Of course, topological spaces is one such model. However, it is not the only one. When working with weak homotopy equivalences there is a Quillen model structure on simplicial sets making it a model for the homotopy category of spaces – in some sense this was one of the examples motivating model categories.

With the definition of a model structure for topological spaces, a natural question was to look for a model where the weak equivalences are h_*-equivalences for some generalised homology theory h_*. This lead Aldridge K. Bousfield to the study of localisations of model categories, which resulted in a new and powerful technique for adding weak equivalences to a model category[Bou75]. Further development by Bousfield, Dror Farjourn and Philip Hirschhorn lead to localisations on a set of morphisms[Hir03].

Using the model structure on simplicial sets as a foundation, André Joyal formulated a model structure on simplicial presheaves of a site in a letter to Alexander Grothendieck in 1983. These ideas were later developed by Rick Jardine[Jar87], and led to model categories becoming an important tool in algebraic geometry. In particular, the $\mathbf{A}^1$-homotopy theory for schemes of Fabien Morel and Vladimir Voevodsky[MV99] is formulated as the Bousfield localisation of a model structure on simplicial presheaves.

Another tool shared by algebraic geometry and algebraic topology is K-theory. This is an invariant that originated in the works of Grothendieck on the Riemann–Roch theorem[BS58]. The definition concerns coherent sheaves on a variety, but a year later Michel F. Atiyah and Friedrich Hirzebruch modified this to work with vector bundles on any finite dimensional CW-complex[AH59]. Moreover, in [AH61] they showed that K-theory is a generalised cohomology theory satisfying Bott periodicity.

Given singular cohomology one can use Poincaré duality to define singular homology for manifolds. The paper [Whi62] by George W. Whitehead showed that there is a similar dual to any generalised cohomology theory. In particular there is a generalised homology theory that is dual to topological K-theory. This theory is called K-homology, and its connection with elliptic operators was noted by Atiyah[Ati70].

The historical recap above dealt with "commutative geometry" in the sense that the spaces can be described by commutative rings. This contrasts to the world of non-commutative geometry where the data is described by C*-algebras. Still, some of the techniques from the commutative setting also works in the non-commutative world. In particular, the Gelfand transform[Gel41] links topological spaces and commutative Banach algebras. This connection is also manifest in K-theory, as the topological K-theory of a space X coincide with the C*-algebra K-theory of $\mathrm{C}(X)$.

The C*-algebra version of K-theory was developed from topological K-theory during the sixties. One step came in 1962 when Richard G. Swan extended a result of Jean-Pierre Serre. The resulting Serre–Swan theorem[Swa62] says that the global section functor gives an equivalence between isomorphism classes of vector bundles over a space X and isomorphism classes of finite dimensional projective modules over the ring $\mathrm{C}(X)$. Since an idempotent in the matrix algebra $\mathrm{Mat}_n\big(\mathrm{C}(X)\big)$ corresponds to a finite dimensional projective module over $\mathrm{C}(X)$, topological K-theory can also be described by projections in the matrix algebras $\mathrm{Mat}_n\big(\mathrm{C}(X)\big)$.

In 1966 Reg Wood proved Bott periodicity by Banach algebra methods[Woo66], but it would still take a decade before K-theory became a mainstream tool in operator algebras. A breakthrough came with the 1975 survey article by Joseph L. Taylor, and its definition of K-theory for Banach algebras[Tay75].

As mentioned earlier, K-homology for spaces is related to elliptic operators. For manifolds these are a particular kind of differential operators, while for more general spaces X (e.g. compact Hausdorff spaces) they are Fredholm operators on the Hilbert space $\mathrm{C}(X)$. Another connection between K-homology and C*-algebras is given by the group $\mathrm{Ext}(X)$ constructed by Lawrence G. Brown, Ronald G. Douglas and Peter A. Fillmore[BDF73]. This group is formed by looking at C*-algebra extensions $\mathbb{K} \to E \to \mathrm{C}(X)$ under a suitable equivalence relation, and by [BDF77] it realises the K-homology of X.

The duality between topological K-theory and K-homology was extended to separable C*-algebras by Gennadi Kasparov in [Kas80]. By using representations on Hilbert modules he defined abelian groups $\mathrm{KK}(A, B)$ such that $\mathrm{KK}(\mathbf{C}, A)$ is the K-theory of A while $\mathrm{KK}(B, \mathbf{C})$ is the K-homology of B. An important aspect of Kasparov's construction is the intersection product $\mathrm{KK}(A, D) \otimes \mathrm{KK}(D, B) \to \mathrm{KK}(A, B)$, which forms the basis of the additive category KK.

In the eighties the axioms for KK-theory were formulated by Nigel Higson[Hig87] and Joachim Cuntz[Cun84], and these axioms played a role in establishing new pictures of $\mathrm{KK}(A, B)$. One such picture, relying on quasi-homomorphisms, is given in [Cun84]. This picture was later modified to use free products instead of quasi-homomorphisms[Cun87]. A related picture, also due to Cuntz, replaces free products with tensor algebras[Cun97] [Cun98]. It is this picture that forms the basis for the construction of KK-theory in the monograph [CMR07], where it is related to the suspension-stable homotopy category $\Sigma\mathrm{Ho}$.

The aforementioned category KK can be viewed as a kind of "stable homotopy category" for C*-algebras. Thus it makes sense to see if there is a corresponding model category. A first step in this direction was in the article [Sch84] where Claude Schochet extended the definition of cofibrations* from topological spaces to C*-algebras. These morphisms can be considered as fibrations, and combined with KK-equivalences as weak equivalences this gives a category of fibrant objects (in the sense of [Bro73])[Uuy11]. However, an argument of Kasper Andersen and Jesper Grodal from 1997 shows that this is *not* the subcategory of fibrant objects in a model category[Uuy11, Appendix A].

Looking back to the commutative side, the model structure used in $\mathbf{A}^1$-homotopy theory for schemes was formed by looking at the category of functors into a "nice" model category and then using Bousfield localisation. In [Øst10] Paul Arne Østvær used this approach to give cubical C*-spaces a model structure related to KK-theory. In view of the Dold–Kan correspondence there ought to be a similar construction where cubical sets are replaced by chain complexes. The aim of this thesis is to investigate such a construction.

Structure of the thesis

Chapter 1 of this thesis deals with background material in homological and homotopical algebra. It contains mostly well-known results, and a reader familiar with chain complexes, triangulated categories and Ab-enriched category theory can skip ahead to Section 1.4. A reader that is also well versed in model categories might want to read the summary of some results from [CD09] in Section 1.4 before proceeding to Chapter 2.

A more detailed description of Chapter 1 is as follows. Section 1.1 deals with notation and terminology and introduces Ab-enriched category theory, Kan extensions, localisations and the categorical "image". The topic of Section 1.2 is the study of the Ab-enriched version of the Yoneda embedding, while Section 1.3 is devoted to triangulated categories.

The aim of Section 1.4 is to introduce a model category structure on the category of chain complexes that is suitable for localisation. Apart from standard works on model categories (i.e. [Hov99] and [Hir03]), this builds upon the work of Denis-Charles Cisinski and Frédéric Déglise on localisations of model categories for chain complexes.

The main part of the thesis is contained in Chapter 2. In Section 2.1 the constructions of Chapter 1 are adapted to the setting of C*-algebras. In order to get a model structure as close as possible to KK-theory, the technique of Bousfield localisation is utilised in Section 2.2. This culminates in a comparison of the resulting homotopy category with the category KK.

Proposition (2.2.15)**.** *The model category constructed in Section 2.2 is such that its homotopy category contains* KK *as a subcategory.*

Section 2.3 deals with the Dold–Kan correspondence between connective chain complexes and simplicial sets. The central result is a Quillen adjunction between the model structure of Section 2.2 and the homotopy invariant model structure on simplicial C*-spaces from [Øst10, Section 3.4]. Following this is Section 2.4 which is devoted to the

* He extended the classical notion of cofibrations, i.e. maps that satisfy the homotopy extension property. These corresponds to the cofibrations in the model structure considered in [Str72], and in this model structure the weak equivalences are homotopy equivalences.

stable situation. Here the main result is the stable analogue of the Dold–Kan correspondence.

Proposition (2.3.14 and 2.4.19)**.** *The Dold–Kan correspondence gives a Quillen adjunction between the model category of Proposition 2.2.15 and the homotopy invariant model structure on simplicial C^*-spaces.*

Similarly, the stable Dold–Kan correspondence gives a Quillen adjunction between the model category of Proposition 2.2.15 and the stable homotopy invariant model structure on spectra of C^*-spaces.

Chapter 2 ends with a section dealing with slice filtrations, which is a computational tool devoted by Voevodsky. The chain complex version of this tool was studied in [HK06] and resulted in a spectral sequence. However, Section 2.5 shows that the spectral sequence obtained from KK does not yield more information.

Lastly, Appendix A deals with the set-theoretic foundations, and is included for completeness.

1 Categorical framework

This chapter introduces background material in homological algebra and category theory. The aim is to present the modern framework for homotopy theory, which later will be used to create a model category structure on C*-algebras.

1.1 Notation and terminology

In this thesis all categories will be locally small, i.e. if C is a category and C_1, C_2 are objects of C then $\mathrm{Hom}_{\mathsf{C}}(C_1, C_2)$ is a set. Note that the category of sets, Set, is a monoidal category with $\times$ as product and the one point set $* = \{\emptyset\}$ as unit. Thus a category C consists of a collection C of objects, and for each pair of objects C_1, C_2 a *set* $\mathrm{Hom}_{\mathsf{C}}(C_1, C_2)$. Moreover, for any triple C_1, C_2, C_3 of objects there is a *function* $\circ_{C_1,C_2,C_3} \colon \mathrm{Hom}_{\mathsf{C}}(C_1, C_2) \times \mathrm{Hom}_{\mathsf{C}}(C_2, C_3) \to \mathrm{Hom}_{\mathsf{C}}(C_1, C_3)$ called composition. Finally, for each object C there is a *function* $\mathrm{id}^C \colon * \to \mathrm{Hom}_{\mathsf{C}}(C, C)$. These functions should satisfy the associativity axiom (i.e. composition is associative) and the unit axiom (i.e. the morphism $\mathrm{id}^C(\emptyset)$ is the identity morphism). Note that both of these axioms can be formulated with diagrams in Set.

Since the category Ab of abelian groups is monoidal, one can mimic the above definition of category by replacing sets with abelian groups, $\times$ with $\otimes$, $*$ with $\mathbf{Z}$ and functions with group homomorphisms. If one does this, the result is the definition of of an Ab-enriched category. Naturally. the prime example of an Ab-enriched category is Ab itself. The Ab-enriched theory goes further. Given two Ab-enriched categories A_1 and A_2 an Ab-functor $\mathrm{F} \colon \mathsf{A}_1 \to \mathsf{A}_2$ consists of for each object $A \in \mathsf{A}_1$ an object $\mathrm{F}(A) \in \mathsf{A}_2$, and for each pair of objects $A, A' \in \mathsf{A}_1$ a group homomorphism $\mathrm{F}_{A,A'} \colon \mathrm{Hom}_{\mathsf{A}_1}(A, A') \to \mathrm{Hom}_{\mathsf{A}_2}(\mathrm{F}(A), \mathrm{F}(A'))$. These group homomorphism should of course respect compositions and identity morphisms.

Natural transformations also have an Ab-enriched version. Given two Ab-functors $\mathrm{F}, \mathrm{G} \colon \mathsf{A}_1 \to \mathsf{A}_2$, an Ab-natural transformation $\eta \colon \mathrm{F} \to \mathrm{G}$ consists of for each object A of A_1 a group homomorphism $\tilde{\eta}_A \colon \mathbf{Z} \to \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A), \mathrm{G}(A)\big)$ such that

$$\begin{tikzcd}
\mathbf{Z} \otimes \mathrm{Hom}_{\mathsf{A}_1}(A, A') \arrow[r, "\tilde{\eta}_A \otimes \mathrm{G}_{A,A'}"] & \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A), \mathrm{G}(A)\big) \otimes \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{G}(A), \mathrm{G}(A')\big) \arrow[d, "\circ"] \\
\mathrm{Hom}_{\mathsf{A}_1}(A, A') \arrow[u, "f \mapsto 1 \otimes f"] \arrow[d, "f \mapsto f \otimes 1"'] & \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A), \mathrm{G}(A')\big) \\
\mathrm{Hom}_{\mathsf{A}_1}(A, A') \otimes \mathbf{Z} \arrow[r, "\mathrm{F}_{A,A'} \otimes \tilde{\eta}_{A'}"] & \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A), \mathrm{F}(A')\big) \otimes \mathrm{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A'), \mathrm{G}(A')\big) \arrow[u, "\circ"']
\end{tikzcd}$$

commutes. Note that since $\operatorname{Hom}_{\mathsf{Ab}}(\mathbf{Z}, G) \simeq G$ for any abelian group G the homomorphism $\tilde{\eta}_A \colon \mathbf{Z} \to \operatorname{Hom}_{\mathsf{A}_2}\big(\mathrm{F}(A), \mathrm{G}(A)\big)$ can be identified with $\eta_A = \tilde{\eta}_A(1) \colon \mathrm{F}(A) \to \mathrm{G}(A)$. Since function composition is bilinear, this simplifies the situation considerably. Notably, the above diagram commutes if and only if

$$\begin{tikzcd} \mathrm{F}(A) \arrow[r, "\eta_A"] \arrow[d, "\mathrm{F}(f)"] & \mathrm{G}(A) \arrow[d, "\mathrm{G}(f)"] \\ \mathrm{F}(A') \arrow[r, "\eta_{A'}"] & \mathrm{G}(A') \end{tikzcd}$$

commutes, that is if η gives a natural transformation $\mathrm{F} \to \mathrm{G}$. To ease the notational complexity, henceforth the data of any Ab-natural transformation between two Ab-functors will be replaced with the data of the corresponding (ordinary) natural transformation. More details on enriched category theory can be found in [Bor94, Chapter 6]. Another notational convention is to denote the set of natural transformations from F to G by $\operatorname{Nat}(\mathrm{F}, \mathrm{G})$.

A particular kind of Ab-enriched categories are the additive categories. Such categories have a zero object 0 and a biproduct $\oplus$. Given two additive categories A_1 and A_2, an additive functor $\mathrm{F} \colon \mathsf{A}_1 \to \mathsf{A}_2$ is an Ab-enriched functor that preserves the additional structure. Thus $\mathrm{F}(0) \simeq 0$ and for each pair of objects A_1, A_2 of A_1 there is an isomorphism $\mathrm{F}(A_1 \oplus A_2) \to \mathrm{F}(A_1) \oplus \mathrm{F}(A_2)$.

There are several places where colimits (also known as "direct limits" or "inductive limits") will be used. In most cases the set-up will be the standard one, where one takes the colimit of a functor $\mathrm{F} \colon \mathsf{C} \to \mathsf{D}$. In notational terms this colimit is denoted by $\operatorname{colim}_{\mathsf{C}} \mathrm{F}$, and F is a C-diagram in D. Note that it is enough to specify the image of a C-diagram in D to determine the value of the colimit (up to a natural isomorphism). Because of this, some colimits will not be specified by a functor $\mathrm{F} \colon \mathsf{C} \to \mathsf{D}$, but will instead be specified by the image of F, and in such cases the terminology of a *diagram in* D will be used.

Among the categories appearing as C in the colimit, the following will often be utilised:

Definition 1.1.1. Let $\mathrm{F} \colon \mathsf{C} \to \mathsf{D}$ be a functor and $D \in \mathsf{D}$. The *category of objects of* C *over* D *(also called the* category of objects F-over D*)*, $\langle \mathrm{F} \downarrow D \rangle$, has as objects pairs (C, f) where C is an object of C and f is a morphism $f \colon \mathrm{F}(C) \to D$ in D, while a morphism $(C_1, f_1) \to (C_2, f_2)$ is a morphism $g \colon C_1 \to C_2$ in C such that

$$\begin{tikzcd} \mathrm{F}(C_1) \arrow[rr, "\mathrm{F}(g)"] \arrow[dr, "f_1"'] & & \mathrm{F}(C_2) \arrow[dl, "f_2"] \\ & D & \end{tikzcd}$$

commutes. There is a corresponding functor $\mathrm{PR} \colon \langle \mathrm{F} \downarrow D \rangle \to \mathsf{C}$ given by sending the object (C, f) to C and the morphism g to g. ♠

One particular use of Definition 1.1.1 is to form Kan extensions. Informally this is a concept in category theory that mimic the process of extending functions by continuity.

Definition 1.1.2. Let $\mathrm{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathrm{G}\colon \mathsf{C} \to \mathsf{E}$ be functors. When it exists, the *left Kan extension of* G *along* F, $\mathrm{Lan}_{\mathrm{F}}\mathrm{G}\colon \mathsf{D} \to \mathsf{E}$, is the functor taking an object D in D to $\mathrm{colim}_{\langle \mathrm{F} \downarrow D\rangle}\, \mathrm{G} \circ \mathrm{PR}$. ♠

The categorically minded reader might notice that this definition of the left Kan extension only covers the pointwise left Kan extension. There is a generalisation which is defined by an universal property[Mac98, Section X.3].

Proposition 1.1.3. *If* E *is cocomplete** *and* $\mathrm{F}\colon \mathsf{C} \to \mathsf{D}$ *is a dense functor*†*, then for any* $\mathrm{G}\colon \mathsf{C} \to \mathsf{E}$ *the functor* $\mathrm{Lan}_{\mathrm{F}}\mathrm{G}\colon \mathsf{D} \to \mathsf{E}$ *exists.*

Proof. Note that for any object D in D, $\langle \mathrm{F} \downarrow D\rangle$ is non-empty. Thus it is enough to look at the effect on morphisms in D. Assume $f\colon D_1 \to D_2$ is a morphism in D and (C, g) is an object of $\langle \mathrm{F} \downarrow D_1\rangle$. In this case $(C, g \circ f)$ is an object of $\langle \mathrm{F} \downarrow D_2\rangle$. This gives the morphism $\mathrm{Lan}_{\mathrm{F}}\mathrm{G}(f)\colon \mathrm{Lan}_{\mathrm{F}}\mathrm{G}(D_1) \to \mathrm{Lan}_{\mathrm{F}}\mathrm{G}(D_2)$. □

The left Kan extension is an extension in the sense of Proposition 1.1.4, whose proof uses the fact that if $C \in \mathsf{C}$ then $(C, \mathrm{id}_{\mathrm{F}(C)})$ is an object of $\langle \mathrm{F} \downarrow \mathrm{F}(C)\rangle$.

Proposition 1.1.4. *Let* E *be a cocomplete category and* $\mathrm{F}\colon \mathsf{C} \to \mathsf{D}$ *a dense functor that is both full and faithful. For any* $\mathrm{G}\colon \mathsf{C} \to \mathsf{E}$ *the induced morphism* $\mathrm{G} \circ \mathrm{PR}\big((C, \mathrm{id}_{\mathrm{F}(C)})\big) \to \mathrm{Lan}_{\mathrm{F}}\mathrm{G}\big(\mathrm{F}(C)\big)$ *gives a natural isomorphism* $\mathrm{G} \to (\mathrm{Lan}_{\mathrm{F}}\mathrm{G}) \circ \mathrm{F}$*.*

Proof. Since F is fully faithful, for each object (C', f) of $\langle \mathrm{F} \downarrow \mathrm{F}(C)\rangle$ there is a morphism $f'\colon C' \to C$ in C such that $\mathrm{F}(f') = f$. It follows that $\mathrm{G}(C)$ has the same universal property as the colimit $\mathrm{Lan}_{\mathrm{F}}\mathrm{G}\big(\mathrm{F}(C)\big)$. □

As mentioned earlier, extensions have parallels to topology. On the other hand the concept of localisation comes from algebra. Recall that when one forms the rational numbers from the integers, one formally inverts every non-zero element. In algebra this process has been generalised to localisation on a multiplicative subsets of a ring (provided the ring has a multiplicative unit). Now, a ring with unit corresponds to a category with one element, where the arrows corresponds to elements in the ring and composition corresponds to multiplication. With this as the basic idea one can also formally invert morphisms in any category:

Definition 1.1.5. Let C be a category and $\mathscr{W}$ a collection of morphisms in C that are closed under composition. The *$\mathscr{W}$-localisation of* C (when it exists) is a category $\mathsf{C}[\mathscr{W}^{-1}]$ and a functor $\mathrm{L}\colon \mathsf{C} \to \mathsf{C}[\mathscr{W}^{-1}]$ satisfying the following universal property: If $\mathrm{F}\colon \mathsf{C} \to \mathsf{D}$ is

* A category E is cocomplete if all small colimits in E exists. † A functor $\mathrm{F}\colon \mathsf{C} \to \mathsf{D}$ is dense if any object D of D can be written as $\mathrm{colim}_{\langle \mathrm{F} \downarrow D\rangle}\, \mathrm{F} \circ \mathrm{PR}$.

a functor that takes morphisms in $\mathscr{W}$ to isomorphisms, then there exist a unique functor $\mathrm{F}'\colon \mathsf{C}[\mathscr{W}^{-1}] \to \mathsf{D}$ such that the diagram

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathrm{F}"] \arrow[dr, "\mathrm{L}"'] & & \mathsf{D} \\ & \mathsf{C}[\mathscr{W}^{-1}] \arrow[ur, "\mathrm{F}'"'] & \end{tikzcd}$$

commutes. ♠

Note that uniqueness of localisations follows from the usual argument with respect to the universal property. Existence of the localisations, however, is a more complicated matter. The construction of localisations follows the pattern of the special case in Example 1.1.6. If the collection $\mathscr{W}$ is not a set this construction might fail on set-theoretic grounds (the hom "sets" in the localised category need not be sets), as Example 1.1.7 shows.

Example 1.1.6. Let C be the category with objects A and B, and the following morphisms

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, bend left, "f"] \arrow[r, bend right, "w"'] & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}$$

Denote by a the symbol $w^{-1}f$ and let $\mathscr{A}$ be the set of all strings (including the empty string) over the alphabet $\{a\}$. Consider the category C' with objects A and B, and where:

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}'}(A, A) &= \{\mathrm{id}_A\} \cup \{w^{-1}fx \mid x \in \mathscr{A}\}, \\ \mathrm{Hom}_{\mathsf{C}'}(B, B) &= \{\mathrm{id}_B\} \cup \{fxw^{-1} \mid x \in \mathscr{A}\}, \\ \mathrm{Hom}_{\mathsf{C}'}(A, B) &= \{w\} \cup \{fx \mid x \in \mathscr{A}\}, \\ \mathrm{Hom}_{\mathsf{C}'}(B, A) &= \{w^{-1}x \mid x \in \mathscr{A}\}. \end{aligned}$$

Composition is given by concatenation subject to the relations $w^{-1}w = \mathrm{id}_A$ and $ww^{-1} = \mathrm{id}_B$. If $\mathscr{W} = \{w\}$, one can show that $\mathrm{C}' \simeq \mathsf{C}[\mathscr{W}^{-1}]$. ♣

Example 1.1.7. For a non-empty set B let $f_B = \{B\}$ and $w_B = \{\{B\}, \{\{B\}\}\}$. Consider the category whose objects are sets, the only endomorphism of an object is the identity, and the only other non-empty hom sets are $\mathrm{Hom}_{\mathsf{C}}(\emptyset, B) = \{f_B, w_B\}$. If $\mathscr{W} = \{w_B \mid B \neq \emptyset\}$ then there is no category $\mathsf{C}[\mathscr{W}^{-1}]$. ♣

Recall that a category A is Ab-enriched (or enriched over Ab) if for objects A_1, A_2 of A the set $\mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$ has the structure of an abelian group and composition is bilinear. In particular, this means that $\mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$ has a zero morphism. To such a category, one can form the category $\mathsf{Ch}(\mathsf{A})$ of chain complexes over A. Its objects are chain complexes $(A_n, d_n)_{n \in \mathbf{Z}}$ where A_n is an object of A and $d_n\colon A_n \to A_{n-1}$ is a morphism in A such that $d_n \circ d_{n+1} = 0$. The morphisms in this category are chain maps, i.e. a morphism $f\colon (A_n, d_n)_{n\in\mathbf{Z}} \to (A'_n, d'_n)_{n\in\mathbf{Z}}$ is given by morphisms $f_n\colon A_n \to A'_n$ in A such that $f_{n-1} \circ d_n = d'_{n-1} \circ f_n$. Since the subscript "$n \in \mathbf{Z}$" on objects makes the notation more cumbersome, it will henceforth be dropped. The resulting category $\mathsf{Ch}(\mathsf{A})$ is then Ab-enriched. However, even more is true. If A is additive, then the category $\mathsf{Ch}(\mathsf{A})$ is additive by defining $(A_n, d_n) \oplus (A'_n, d'_n) = (A_n \oplus A'_n, d_n \oplus d'_n)$.

The next definition has its genesis in topology. Recall that the reduced suspension of a based topological space X is $\Sigma X = \mathbf{S}^1 \wedge X$. It follows that if X is a based CW-complex, then its n-cells are in bijective correspondence* with the $n+1$-cells of ΣX. Thus if one looks at the cellular chain complex of ΣX, it is a "shifted" version of the cellular chain complex of X. Similarly, if $f\colon X \to Y$ is a base-point preserving continuous map, then one can form the mapping cone, $\mathrm{cone}\,(f)$, by "glueing the cone of X to Y along f". In this case the $n+1$-cells of $\mathrm{cone}\,(f)$ comes from two sources, the $n+1$-cells of Y and the n-cells of X. These two sources of cells are connected by f, which connects parts of the boundary of cells from the cone of X with cells from Y.

Definition 1.1.8. Let C be an additive category.

- The *suspension* functor (also called the shift functor or the translation functor) on the category $\mathsf{Ch}(\mathsf{C})$ is the invertible endofunctor $\Sigma\colon \mathsf{Ch}(\mathsf{C}) \to \mathsf{Ch}(\mathsf{C})$ given by $(C_n, d_n) \mapsto (\hat{C}_n, \hat{d}_n)$ where $\hat{C}_n = C_{n-1}$ and $\hat{d}_n = d_{n-1}$. It has the obvious effect on morphisms.
- If $f\colon (C_n, d_n) \to (C'_n, d'_n)$ is a chain map, then the *mapping cone* of f is
$$\mathrm{cone}\,(f) = \left(C_{n-1} \oplus C'_n, \begin{pmatrix} -d_{n-1} & 0 \\ -f_{n-1} & d'_n \end{pmatrix}\right).$$
- A chain map $f\colon (C_n, d_n) \to (C'_n, d'_n)$ is *null-homotopic* if there are morphisms $s_n\colon C_n \to C'_{n+1}$ in C such that $f_n = s_{n-1} \circ d_n + d'_{n+1} \circ s_n$.
- A chain complex (C_n, d_n) is *contractible* if $\mathrm{id}_{(C_n, d_n)}$ is null-homotopic. ♠

If $f\colon (C_n, d_n) \to (C'_n, d'_n)$ is a chain map, then the mapping cone of f and the suspension of C is related by the sequence
$$(C_n, d_n) \xrightarrow{f} (C'_n, d'_n) \xrightarrow{\iota_f} \mathrm{cone}\,(f) \xrightarrow{\pi_f} \Sigma(C'_n, d'_n)$$
with ι_f and π_f the obvious chain maps. Note that the chain map $\pi_f \circ \iota_f$ is the zero chain map, while $\iota_f \circ f$ is null-homotopic.

The above sequence also has its genesis in topology. Let $f\colon X \to Y$ be a basepoint preserving continuous map, and note that there is an obvious inclusion $i\colon Y \hookrightarrow \mathrm{cone}\,(f)$. Then $i \circ f$ is homotopic to a map that sends everything to the base-point of CX. There is also a continuous map $p\colon \mathrm{cone}\,(f) \to \Sigma X$ obtained by collapsing Y to the basepoint, and the composition $p \circ i$ obviously maps everything to this point.

* At least if $n \geq 0$ and the basepoint is considered as a -1-cell.

If $(\mathsf{C}, \otimes, S)$ is a symmetric monoidal category with C additive, then the total tensor product gives $\mathsf{Ch}(\mathsf{C})$ a symmetric monoidal category structure. For objects (C_n, d_n), (C'_n, d'_n) in $\mathsf{Ch}(\mathsf{C})$ the total tensor product is given by

$$(C_n, d_n) \otimes_T (C'_n, d'_n) = \Big(\bigoplus_{i+j=n} C_i \otimes C'_j, \bigoplus_{i+j=n} \Big(\begin{smallmatrix} d_i \otimes \mathrm{id}_{C'_j} \\ (-1)^i \mathrm{id}_{C_i} \otimes d'_j \end{smallmatrix} \Big) \Big).$$

The unit for the total tensor product is the chain complex $S^0 = \mathbf{S}^0(S)$ that is S in degree 0 and 0 elsewhere, while the twist isomorphism $(C_n, d_n) \otimes_T (C'_n, d'_n) \to (C'_n, d'_n) \otimes_T (C_n, d_n)$ takes $c \otimes c' \in C_i \otimes C'_j$ to $(-1)^{ij} c' \otimes c \in C'_j \otimes C_i$.

A particular class of symmetric monoidal categories are the ones that are also closed monoidal, i.e. there is a right adjoint to the functor $C \otimes \_$. Such a functor is called an internal hom, and denoted by $\underline{\mathrm{Hom}}(C, \_)$. In the special case where $\mathsf{C} = \mathsf{Ab}$, the internal hom object is $\mathrm{Hom}_{\mathsf{Ab}}(C, \_)$, and in this case there is also an internal hom object in the category $\mathsf{Ch}(\mathsf{Ab})$.

Definition 1.1.9. For chain complexes (C_n, d_n), (C'_n, d'_n) in $\mathsf{Ch}(\mathsf{Ab})$, the *internal hom chain complex* $\underline{\mathrm{Hom}}\Big((C_n, d_n), (C'_n, d'_n)\Big)$ is

$$\underline{\mathrm{Hom}}\big((C_n, d_n), (C'_n, d'_n)\big) = \Big(\prod_l \mathrm{Hom}_{\mathsf{Ab}}(C_l, C'_{l+n}), D_n \Big),$$

where

$$D_n \colon \prod_l \mathrm{Hom}_{\mathsf{Ab}}(C_l, C'_{l+n}) \to \prod_k \mathrm{Hom}_{\mathsf{Ab}}(C_k, C'_{k+n-1}), \quad (f_l) = f \mapsto D_n f$$

and $D_n f$ is the map whose kth component is

$$(D_n f)_k \colon C_k \to C'_{k+n-1}, \quad c \mapsto (-1)^{n-1} f_{k-1} \circ d_k(c) + d'_{k+n} \circ f_k(c). \qquad \spadesuit$$

In this case, the hom chain complex makes $\mathsf{Ch}(\mathsf{Ab})$ a closed symmetric monoidal category, so for chain complexes C, C', and C'' there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ch}(\mathsf{Ab})}\Big(C, \underline{\mathrm{Hom}}(C', C'')\Big) \to \mathrm{Hom}_{\mathsf{Ch}(\mathsf{Ab})}\Big(C \otimes_T C', C''\Big).$$

For a general additive closed* category C, one use the internal hom object of C to endow $\mathsf{Ch}(\mathsf{C})$ with an internal hom object following a similar recipe as the one of Definition 1.1.9.

A final observation is that in degree n the hom chain complex is linked to suspension n times. One formulation that makes this link explicit is Remark 1.1.10, which relies on the functor that forgets differentials.

* A category is closed if it has an internal hom object.

Remark 1.1.10. Denote by $\mathsf{Ab}^{\mathbf{Z}}$ the category of functors from $\mathbf{Z}$ (where $\mathbf{Z}$ is viewed as a discrete category, i.e. it has only the identity morphisms) to Ab with natural transformations as morphisms, and let $\mathsf{O}\colon \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{Ab}^{\mathbf{Z}}$ be the functor that forgets the differentials. For chain complexes (C_n, d_n), (C'_n, d'_n) in $\mathsf{Ch}(\mathsf{Ab})$, there is a natural isomorphism

$$\begin{aligned}\underline{\mathrm{Hom}}\big((C_n, d_n), (C'_n, d'_n)\big)_i &\to \mathrm{Hom}_{\mathsf{Ab}^{\mathbf{Z}}}\Big(\mathrm{O}\big((C_n, d_n)\big), \mathrm{O} \circ \Sigma^{-i}\big((C'_n, d'_n)\big)\Big)\\ (f_l) &\mapsto f.\end{aligned}$$ ■

One aspect of a chain complex is its homology, i.e. the result of using the functors $\mathrm{H}_k\colon \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{Ab}$ given by $\mathrm{H}_k\big((C_n, d_n)\big) = \ker d_k / \operatorname{im} d_{k+1}$. Later in the thesis there will be a need to compute the homology of chain complexes of functors. In order to do so the following definitions of the kernel and image are useful:

Definition 1.1.11. In a pointed category C, the *kernel* of a morphism $f\colon C_1 \to C_2$ is the equalizer of $C_1 \underset{0}{\overset{f}{\rightrightarrows}} C_2$. Thus the kernel of f is a pair $(\ker f, \iota)$ where $\iota\colon \ker f \to C_1$ is a morphism of C such that

- the composition $f \circ \iota$ factors through 0 and
- if $g\colon C_0 \to C_1$ is such that $f \circ g$ factors through 0, then g factors uniquely through ι. ♠

Note that by the second bullet point, the map $\iota\colon \ker f \to C_1$ is necessarily a monomorphism.

Definition 1.1.12. Let C be a category with terminal object $*$, and let $f\colon C_1 \to C_2$ be a morphism in C. The *image* of f is a triple $(\operatorname{im} f, \pi, \iota)$ where $\pi\colon C_1 \to \operatorname{im} f$ and $\iota\colon \operatorname{im} f \to C_2$ are morphisms in C such that $f = \iota \circ \pi$. Such a triple should satisfy

- if $h\colon C_2 \to C_3$ is a morphism such that $h \circ f$ factors through $*$, then $h \circ \iota$ also factors through $*$,
- the morphism ι is a monomorphism, and
- the triple $(\operatorname{im} f, \pi, \iota)$ is terminal among such triples. ♠

Here, "terminal among such triples" means that if another triple (T, p, i) with $i \circ p = f$ satisfy the two first bullet-points then there exists a unique morphism $t\colon T \to \operatorname{im} f$ such that the diagram

$$\begin{tikzcd} & T \arrow[d, "t"] \arrow[dr, "i"] & \\ C_1 \arrow[ur, "p"] \arrow[r, "\pi"'] & \operatorname{im} f \arrow[r, "\iota"'] & C_2 \end{tikzcd}$$

commutes.

Lemma 1.1.13. *Let C be category with a terminal object, all push-outs and all equalizers. Then all morphisms of C have an image.*

Proof. Given $f\colon C_1 \to C_2$, denote by $C_2/\operatorname{im} f$ the push-out

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner."] & * \arrow[d, "b"] \\
C_2 \arrow[r, "g"] & C_2/\operatorname{im} f.
\end{tikzcd}$$

Let $a\colon C_2 \to *$ be the unique morphism to the terminal object, and observe that $g \circ f = b \circ a \circ f$. Consider the equalizer (K, ι) of g and $b \circ a$, and note that there is a morphism $\pi\colon C_1 \to K$ such that $f = \iota \circ \pi$. Moreover, if $h\colon C_2 \to C_3$ is such that $h \circ f$ factors through $*$, then by the universal property of the push-out there is a morphism $k\colon C_2/\operatorname{im} f \to C_3$ such that $h = k \circ g$. It follows that $h \circ \iota = k \circ g \circ \iota$, and the latter factors through $*$ since $g \circ \iota = b \circ a \circ \iota$. Thus the triple (K, π, ι) is the image of f if it is terminal among such triples. If (T, p, i) is another such triple, then the diagram

$$\begin{tikzcd}
C_1 \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[dr, "f"] & T \arrow[r] \arrow[d, "i"] & * \arrow[d, "b"] \\
K \arrow[r, "\iota"] & C_2 \arrow[r, "g"] \arrow[ur, "a"] & C_2/\operatorname{im} f
\end{tikzcd}$$

commutes. By the universal property of equalizers there is a unique morphism $t\colon T \to K$ such that $\iota \circ t = i$. Moreover $\iota \circ \pi = i \circ p = \iota \circ t \circ p$, so $\pi = t \circ p$ since ι is a monomorphism. Thus (K, π, ι) is the image of f. $\square$

Lemma 1.1.14. *If $f\colon C_1 \to C_2$ and $h\colon C_2 \to C_3$ are morphisms of a pointed category C such that $h \circ f$ factors through 0, h has a kernel and f has an image, then there is a unique monomorphism $\alpha\colon \operatorname{im} f \to \ker h$.*

Proof. Since $h \circ f = 0$, the following diagram commutes

$$\begin{tikzcd}
C_1 \arrow[r, "\pi"] \arrow[d] & \operatorname{im} f \arrow[r, "\iota"] & C_2 \arrow[d, "h"] \\
0 \arrow[rr] & & C_3.
\end{tikzcd}$$

Thus $h \circ \iota = 0$, whence the universal property of kernels gives a unique monomorphism $\alpha\colon \operatorname{im} f \to \ker h$. $\square$

1.2 The Ab-enriched Yoneda embedding

If A is an Ab-enriched category then one can view $\operatorname{Hom}_{\mathsf{A}}$ as a bifunctor (Ab-enriched in each variable) from $\mathsf{A}^{\mathrm{op}} \times \mathsf{A}$ to Ab. This gives rise to the Yoneda embedding Y of A^{op} into the category $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ of Ab-enriched functors from A to Ab (with Ab-natural transformations as morphisms). Under this embedding, the object A is mapped to $\operatorname{Hom}_{\mathsf{A}}(A, \_)$ while the morphism $f\colon A \to A'$ is mapped to the natural transformation

$f^*\colon \mathrm{Hom}_{\mathsf{A}}(A', \_) \to \mathrm{Hom}_{\mathsf{A}}(A, \_)$. By the Ab-enriched Yoneda lemma* this embedding is a fully faithful functor.

Since Ab is an additive category, so is $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ by defining $(\mathrm{F} \oplus \mathrm{G})(A) = \mathrm{F}(A) \oplus \mathrm{G}(A)$. Moreover, since $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ is complete and cocomplete, it has all kernels and cokernels. Thus $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ is an abelian category. It is also possible to show that $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ is a Grothendieck category, see for instance [Fre03, The proof of Proposition 5.21].

The next aim is to give $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ the structure of a closed monoidal category. Since Ab has a tensor product, the first candidate is to define the hom object by $\mathrm{Hom}(\mathrm{G}, \mathrm{H})A = \mathrm{Nat}\big(\mathrm{G}(\_), \mathrm{H}(A) \otimes \mathrm{H}(\_)\big)$ and use the *external tensor product* in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$, that is the functor

$$\begin{aligned} \_ \otimes \_ \colon [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} \times [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} &\to [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} \\ (\mathrm{F}_1 \otimes \mathrm{F}_2)(A) &= \mathrm{F}_1(A) \otimes \mathrm{F}_2(A). \end{aligned}$$

The problem with the external tensor product and the above hom object is that the hom–tensor adjunction fails.

Example 1.2.1. Let $\mathrm{F}, \mathrm{G}, \mathrm{H} \in [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ and $\eta\colon \mathrm{F} \to \mathrm{Hom}(\mathrm{G}, \mathrm{H})$. If the hom–tensor adjunction holds, η should give rise to a natural transformation $\theta\colon \mathrm{F} \otimes \mathrm{G} \to \mathrm{H}$. Assume F and G both sends all morphisms to $\mathrm{id}_{\mathbf{Z}}$ while H is functor that maps all morphisms to $\mathrm{id}_{\mathbf{Z}^2}$. In this case η is determined by $\eta_A\colon \mathbf{Z} \to \mathrm{Nat}\big(\mathrm{G}(\_), \mathrm{H}(A) \otimes \mathrm{H}(\_)\big)$ which again is determined by the value of $\eta_A(1)\colon \mathrm{G} \to \mathrm{H}(A) \otimes \mathrm{H}$, and the latter natural transformation is determined by the value of $\big(\eta_A(1)\big)_A(1) \in \mathbf{Z}^2 \otimes \mathbf{Z}^2$. On the other hand θ_A is uniquely determined by the value of $\theta_A(1 \otimes 1) \in \mathrm{H}(A) = \mathbf{Z}^2$. Since there is no canonical map $\mathbf{Z}^2 \otimes \mathbf{Z}^2 \to \mathbf{Z}^2$, this shows that the hom–tensor adjunction fails. ♣

If A is a monoidal category, there is a construction of internal objects due to Brian Day ([Day70]). A reformulation of this construction can be found in [MMSS01, Section 21]. The technique is to use functor

$$\begin{aligned} \_ \otimes' \_ \colon [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} \times [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} &\to [\mathsf{A} \times \mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}} \\ (\mathrm{F}_1 \otimes' \mathrm{F}_2)(A_1, A_2) &= \mathrm{F}_1(A_1) \otimes \mathrm{F}_2(A_2) \end{aligned}$$

to internalise the external tensor product, and then modify $\mathrm{Hom}(\mathrm{G}, \mathrm{H})$ in the appropriate way.

Definition 1.2.2. If A is monoidal with tensor product $\_\otimes_\sigma\_\colon \mathsf{A} \times \mathsf{A} \to \mathsf{A}$, then the *internal tensor product* in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ of the functors F_1 and F_2 is the functor $(\mathrm{F}_1 \underline{\otimes} \mathrm{F}_2) = \mathrm{Lan}_{\otimes_\sigma}(\mathrm{F}_1 \otimes' \mathrm{F}_2)$, the left Kan extension† of $\mathrm{F}_1 \otimes' \mathrm{F}_2$ along $\otimes_\sigma$. Thus for A an object of A,

$$(\mathrm{F}_1 \underline{\otimes} \mathrm{F}_2)(A) = \operatorname*{colim}_{\langle \otimes_\sigma \downarrow A \rangle} (\mathrm{F}_1 \otimes' \mathrm{F}_2) \circ \mathrm{PR} = \operatorname*{colim}_{A_1 \otimes_\sigma A_2 \to A} \mathrm{F}_1(A_1) \otimes \mathrm{F}_2(A_2).$$

Note that another name for the internal tensor product, $\underline{\otimes}$, is the Day convolution product. ♠

---

* This is the Yoneda lemma where everything is Ab-enriched, and the functors go to Ab. The proof of this lemma is the same as that of the Yoneda lemma, mutatis mutandis. † Which exists due to Theorem 1.2.4.

The internal tensor product, $\underline{\otimes}$, is an extension of $\otimes_\sigma$ in the sense that if B_1 and B_2 are objects of A then the natural map $\mathrm{Y}(B_1)\underline{\otimes}\mathrm{Y}(B_2) \to \mathrm{Y}(B_1 \otimes_\sigma B_2)$ is an isomorphism [MMSS01, Lemma 1.8]. Moreover, if A is a symmetric monoidal category, then so is $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ with the internal tensor product.

Note that if $\mathrm{H} \in [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$, then there is no guarantee that $\mathrm{H}(A_1 \otimes_\sigma A_2)$ and $\mathrm{H}(A_1) \otimes \mathrm{H}(A_2)$ are isomorphic. Since the internal tensor product involves morphisms $A_1 \otimes_\sigma A_2 \to A$ in A one must account for this in the definition if the internal hom object.

Definition 1.2.3. Let A be a monoidal with tensor product $\otimes_\sigma$. For functors G, H in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ the *internal hom object* is defined by

$$\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A = \mathrm{Nat}\left(\mathrm{G}(\_), \mathrm{H}(A \otimes_\sigma \_)\right)$$

for objects A in A, and

$$\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})f \colon \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})B, \quad \eta \mapsto \left(\mathrm{H}(f \otimes_\sigma \mathrm{id}\_)\right) \circ \eta$$

for morphisms $f \colon A \to B$ in A. ♠

By construction, these two internal objects satisfy the usual adjoint relation, that is for functors $\mathrm{F}, \mathrm{G}, \mathrm{H} \in [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ there is a natural isomorphism

$$\mathrm{Nat}\Big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})\Big) \to \mathrm{Nat}\Big(\mathrm{F}\underline{\otimes}\mathrm{G}, \mathrm{H}\Big).$$

Next up is a process for extending functor from A to $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$. In order to do this the following two density results are needed:

Theorem 1.2.4. *Any object* $\mathrm{F} \in [\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ *is the colimit* $\mathrm{colim}_{\langle \mathrm{Y} \downarrow \mathrm{F} \rangle}(\mathrm{Y} \circ \mathrm{PR})$ *of representable objects (i.e. the Yoneda embedding is dense).*

Proposition 1.2.5. *Any object* β *in the arrow category of* $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ *(i.e. any* Ab*-natural transformation) is a colimit of representable objects.*

Proof of Theorem 1.2.4. Let $\mathrm{F} \colon \mathsf{A} \to \mathsf{Ab}$ be an Ab-functor and A an object of A. By the Ab-enriched Yoneda lemma there is an isomorphism (natural in both A and F) of abelian groups

$$\mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A, \_), \mathrm{F}\right) \simeq \mathrm{F}(A), \quad \eta \mapsto \eta_A(\mathrm{id}_A),$$

where the group structure on $\mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A, \_), \mathrm{F}\right)$ is given by pointwise addition. The inverse of this isomorphism is given by $x \mapsto \eta^x$ where

$$\eta^x_B \colon \mathrm{Hom}_{\mathsf{A}}(A, B) \to \mathrm{F}(B), \quad k \mapsto (\mathrm{F}k)(x).$$

Taking this as a starting point, the goal is to construct a diagram in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ consisting of objects and morphisms in the image of Y such that F is the colimit of this diagram.

So given $A \in \mathsf{A}$ and $x \in \mathrm{F}(A)$ let the diagram have a vertex $\mathrm{Hom}_{\mathsf{A}}(A, \_)_x$. Moreover, for each morphism $f \colon A \to A'$ in A and $x \in \mathrm{F}(A)$ let the diagram have an arrow

$\zeta^{f,x}\colon \mathrm{Hom}_{\mathsf{A}}(A', \_)_{(\mathrm{F}f)(x)} \to \mathrm{Hom}_{\mathsf{A}}(A, \_)_x$. The interpretation of this diagram is straightforward. The vertices $\mathrm{Hom}_{\mathsf{A}}(A, \_)_x$ are all copies of the object $\mathrm{Hom}_{\mathsf{A}}(A, \_)$, while the arrows $\zeta^{f,x}$ are morphisms in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ (i.e. Ab-natural transformations) determined by

$$\zeta^f_B\colon \mathrm{Hom}_{\mathsf{A}}(A', B) \to \mathrm{Hom}_{\mathsf{A}}(A, B), \quad k' \mapsto f^*(k') = k' \circ f.$$

For book-keeping reasons the objects and morphisms obtained from the diagram will be adorned with a subscript or superscript indicating which vertex or arrow they came from.

Claim: The colimit of the diagram is the functor F

There are two things to check in order to verify the claim:

1. The existence of a morphism from the colimit to the functor F.
2. The morphism is in fact an isomorphism.

Proof of (1): Given an object A in A, for each $x \in \mathrm{F}(A)$ and for each object B in A define the map

$$\eta^x_B\colon \mathrm{Hom}_{\mathsf{A}}(A, B)_x \to \mathrm{F}(B), \quad k \mapsto (\mathrm{F}k)(x).$$

Since $\mathrm{F}k$ is a group homomorphism and $\eta^x_B(k)$ is the evaluation of this group homomorphism at x, it follows that $\eta^x_B(k)$ itself is a group homomorphism.

Now if $g\colon B \to C$ is a morphism in A, then

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{A}}(A, B)_x \arrow[r, "\eta^x_B"] \arrow[d, "g_*"] & \mathrm{F}(B) \arrow[d, "\mathrm{F}g"] \\
\mathrm{Hom}_{\mathsf{A}}(A, C)_x \arrow[r, "\eta^x_C"] & \mathrm{F}(C)
\end{tikzcd}$$

commutes since

$$(\mathrm{F}g) \circ \eta^x_B(k) = (\mathrm{F}g)\big((\mathrm{F}k)(x)\big) = \big(\mathrm{F}(g \circ k)\big)(x) = \eta^x_C \circ g_*(k).$$

Thus the maps η^x_B form an Ab-natural transformation $\eta^x\colon \mathrm{Hom}_{\mathsf{A}}(A, \_)_x \to \mathrm{F}$, and evidently $\eta^x_A(\mathrm{id}_A) = x$. Moreover, if $\zeta^{f,x}\colon \mathrm{Hom}_{\mathsf{A}}(A', \_)_{(\mathrm{F}f)(x)} \to \mathrm{Hom}_{\mathsf{A}}(A, \_)_x$ is an arrow in the diagram and $k'\colon A' \to B$ is a morphism in A, then

$$\eta^x_A \circ \zeta^{f,x}(k') = \eta^x_A(k' \circ f) = \big(\mathrm{F}(k' \circ f)\big)(x) = (\mathrm{F}k')\big((\mathrm{F}f)(x)\big) = \eta^{(\mathrm{F}f)(x)}_{A'}(k').$$

It follows that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}f)(x)} \arrow[rr, "\zeta^{f,x}"] \arrow[dr, "\eta^{(\mathrm{F}f)(x)}"'] & & \mathrm{Hom}_{\mathsf{A}}(A, _)_x \arrow[dl, "\eta^x"] \\
& \mathrm{F} &
\end{tikzcd}$$

commutes.

Let

$$\theta^x\colon \mathrm{Hom}_{\mathsf{A}}(A, \_)_x \to \operatorname*{colim}_{\langle \mathrm{Y} \downarrow \mathrm{F} \rangle}(\mathrm{Y} \circ \mathrm{PR})$$

be the maps given by the colimit. By the universal property of colimits there is a natural transformation

$$\xi\colon \operatorname*{colim}_{\langle \mathrm{Y}\downarrow \mathrm{F}\rangle}(\mathrm{Y}\circ \mathrm{PR}) \to \mathrm{F}$$

such that $\xi \circ \theta^x = \eta^x$.

Proof of (2): Since $\eta^x(\mathrm{id}_A) = x$, ξ_A is surjective for all A. Thus it remains to show that ξ_A is injective. Since the colimit is computed pointwise,

$$\big(\operatorname*{colim}_{\langle \mathrm{Y}\downarrow \mathrm{F}\rangle}(\mathrm{Y}\circ \mathrm{PR})\big)(B) = \{(x, k\colon A \to B) \mid x \in \mathrm{F}(A)\}/\sim$$

where $(x,k) \sim (x',k')$ if there is a morphism $f\colon A \to A'$ such that $(\mathrm{F}f)(x) = x'$ and $k = k' \circ f$. Moreover, $\theta^x_B\colon \mathrm{Hom}_{\mathsf{A}}(A,B)_x \to \big(\mathrm{colim}_{\langle \mathrm{Y}\downarrow \mathrm{F}\rangle}(\mathrm{Y}\circ \mathrm{PR})\big)(B)$ is given by $k \mapsto \langle x,k\rangle$. Clearly $(x,k) \sim \big((\mathrm{F}k)(x), \mathrm{id}_B\big)$ and $\xi_B(\langle y, \mathrm{id}_B\rangle) = \xi_B \circ \theta^y_B(\mathrm{id}_B) = \eta^y_B(\mathrm{id}_B) = y$ for $y \in \mathrm{F}(B)$. Thus if $\xi_B(\langle x,k\rangle) = \xi_B(\langle x',k'\rangle)$ then $\xi_B\big(\langle (\mathrm{F}k)(x), \mathrm{id}_B\rangle\big) = \xi_B\big(\langle (\mathrm{F}k')(x'), \mathrm{id}_B\rangle\big)$ whence $(\mathrm{F}k)(x) = (\mathrm{F}k')(x')$.

Concluding remark: This finishes the proof of the "representable objects" part of Theorem 1.2.4. Note that by the Yoneda lemma

$$\mathrm{Nat}\,\big(\mathrm{Hom}_{\mathsf{A}}(A, \_), \mathrm{F}\big) \simeq \mathrm{F}(A), \quad \eta \mapsto \eta_A(\mathrm{id}_A)$$

so the vertices of the diagram are in one-to-one correspondence with the objects of the category $\langle \mathrm{Y} \downarrow \mathrm{F}\rangle$ of objects Y-over F. Similarly there is a one-to-one correspondence between the arrows in the diagram and the morphisms of the category $\langle \mathrm{Y} \downarrow \mathrm{F}\rangle$. ☐

The proof of Proposition 1.2.5 needs the following lemma, whose proof follows from the fact that every abelian group has a zero element.

Lemma 1.2.6. *Let* C *be an* Ab*-enriched category and* $\mathrm{F}\colon \mathsf{B} \to \mathsf{C}$ *some functor. If* B *is a subcategory of* B'*, consider the extension of* F *given by* $\mathrm{F}'\colon \mathsf{B}' \to \mathsf{C}$,

$$\mathrm{F}'(B) = \begin{cases} \mathrm{F}(B) & B \textit{ is an object of } \mathsf{B} \\ 0 & \textit{otherwise,} \end{cases} \qquad \mathrm{F}'(f) = \begin{cases} \mathrm{F}(f) & f \textit{ is a morphism of } \mathsf{B} \\ 0 & \textit{otherwise.} \end{cases}$$

In this case, if the colimit $\mathrm{colim}_{\mathsf{B}}\,\mathrm{F}$ *exists, so does the colimit* $\mathrm{colim}_{\mathsf{B}'}\,\mathrm{F}'$*. Moreover* $\mathrm{colim}_{\mathsf{B}}\,\mathrm{F}$ *and* $\mathrm{colim}_{\mathsf{B}'}\,\mathrm{F}'$ *are isomorphic.* ☐

Proof of Proposition 1.2.5. Let $\mathrm{F}, \mathrm{F}'\colon \mathsf{A}^{\mathrm{op}} \to \mathsf{Ab}$ be Ab-functors and $\beta\colon \mathrm{F} \to \mathrm{F}'$ a natural transformation. The goal is to define a diagram ‡ whose colimit is the natural transformation β. As a beginning, the proof of Theorem 1.2.4 gives us a "subdiagram" † constructed as follows:

Given $A \in \mathsf{A}$ and $x \in \mathrm{F}(A)$ let the diagram † have a vertex $\mathrm{Hom}_{\mathsf{A}}(A, \_)_x$, and for each morphism $f\colon A \to A'$ in A and $x \in \mathrm{F}(A)$ let the diagram † have an arrow $\zeta^{f,x}_{\square}\colon \mathrm{Hom}_{\mathsf{A}}(A', \_)_{(\mathrm{F}f)(x)} \to \mathrm{Hom}_{\mathsf{A}}(A, \_)_x$. The interpretation of this diagram is a bit more complicated than in the proof of Theorem 1.2.4. In this case the vertex $\mathrm{Hom}_{\mathsf{A}}(A, \_)_x$ maps to the object

$$\mathrm{Hom}_{\mathsf{A}}(A, \_)_x \xrightarrow{\iota^x} \mathrm{Hom}_{\mathsf{A}}(A, \_)_{\beta_A(x)}$$

(as in the proof of Theorem 1.2.4, the "decorations" are here for book-keeping reasons) with ι^x the identity natural transformation, while the arrow $\zeta^{f,x}_{\square}\colon \operatorname{Hom}_{\mathsf{A}}(A', \_)_{(\mathrm{F}f)(x)} \to \operatorname{Hom}_{\mathsf{A}}(A, \_)_x$ maps to the morphism (i.e. the commutative diagram)

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}f)(x)} \arrow[r, "\zeta^{f,x}"] \arrow[d, "\iota^{(\mathrm{F}f)(x)}"] & \operatorname{Hom}_{\mathsf{A}}(A, _)_x \arrow[d, "\iota^x"] \\
\operatorname{Hom}_{\mathsf{A}}(A', _)_{\beta_{A'} \circ (\mathrm{F}f)(x)} \arrow[r, "\zeta^{f,\beta_A(x)}"] & \operatorname{Hom}_{\mathsf{A}}(A, _)_{\beta_A(x)}
\end{tikzcd}$$

where $\zeta^{f,x}$ and $\zeta^{f,\beta_A(x)}$ both are the natural transformation given by

$$\zeta^f_B\colon \operatorname{Hom}_{\mathsf{A}}(A', B) \to \operatorname{Hom}_{\mathsf{A}}(A, B), \quad k' \mapsto f^*(k') = k' \circ f.$$

For $x \in \mathrm{F}(A)$ define the natural transformation $\eta^x\colon \operatorname{Hom}_{\mathsf{A}}(A, \_)_x \to \mathrm{F}$ by

$$\eta^x_B\colon \operatorname{Hom}_{\mathsf{A}}(A, B)_x \to \mathrm{F}(B), \quad k \mapsto (\mathrm{F}k)(x)$$

i.e. η^x is defined identical to its namesake in the proof of Theorem 1.2.4. Similarly, for $x' \in \mathrm{F}'(A)$ define the natural transformation $\eta'^{x'}\colon \operatorname{Hom}_{\mathsf{A}}(A, \_)_{x'} \to \mathrm{F}'$ by

$$\eta'^{x'}_B\colon \operatorname{Hom}_{\mathsf{A}}(A, B)_{x'} \to \mathrm{F}'(B), \quad k \mapsto (\mathrm{F}'k)(x').$$

Observe that if $k \in \operatorname{Hom}_{\mathsf{A}}(A, B)$, then

$$\beta_B \circ \eta^x_B(k) = \beta_B \circ (\mathrm{F}k)(x) = (\mathrm{F}'k) \circ \beta_A(x) = \eta'^{\beta_A(x)}_B(k) = \eta'^{\beta_A(x)}_B \circ \iota^x_A(k)$$

so the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}f)(x)} \arrow[rr, "\zeta^{f,x}"] \arrow[dr, "\eta^{(\mathrm{F}f)(x)}"'] \arrow[ddd, "\iota^{(\mathrm{F}f)(x)}"'] & & \operatorname{Hom}_{\mathsf{A}}(A, _)_x \arrow[dl, "\eta^x"] \arrow[ddd, "\iota^x"] \\
& \mathrm{F} \arrow[d, "\beta"] & \\
& \mathrm{F}' & \\
\operatorname{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}'f) \circ \beta_A(x)} \arrow[rr, "\zeta^{f,\beta_A(x)}"] \arrow[ur, "\eta'^{(\mathrm{F}'f)(\beta_A(x))}"] & & \operatorname{Hom}_{\mathsf{A}}(A, _)_{\beta_A(x)} \arrow[ul, "\eta'^{\beta_A(x)}"']
\end{tikzcd}$$

commutes. Moreover, given Ab-functors $\mathrm{G}, \mathrm{G}'\colon \mathsf{A} \to \mathsf{Ab}$, a natural transformation $\pi\colon \mathrm{G} \to \mathrm{G}'$, and natural transformations θ, θ' such that

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}f)(x)} \arrow[rr, "\zeta^{f,x}"] \arrow[dr, "\theta^{(\mathrm{F}f)(x)}"'] \arrow[ddd, "\iota^{(\mathrm{F}f)(x)}"'] & & \operatorname{Hom}_{\mathsf{A}}(A, _)_x \arrow[dl, "\theta^x"] \arrow[ddd, "\iota^x"] \\
& \mathrm{G} \arrow[d, "\pi"] & \\
& \mathrm{G}' & \\
\operatorname{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}'f) \circ \beta_A(x)} \arrow[rr, "\zeta^{f,\beta_A(x)}"] \arrow[ur, "\theta'^{(\mathrm{F}'f)(\beta_A(x))}"] & & \operatorname{Hom}_{\mathsf{A}}(A, _)_{\beta_A(x)} \arrow[ul, "\theta'^{\beta_A(x)}"']
\end{tikzcd}$$

commutes, let

$$\xi\colon \mathrm{F} \to \mathrm{G}, \quad \xi_A(x) = \theta_A^x(\mathrm{id}_A) \qquad \text{and} \qquad \xi'\colon \mathrm{F}' \to \mathrm{G}', \quad \xi'_A(y) = \theta_A'^y(\mathrm{id}_A).$$

It follows that

$$\pi_A \circ \xi_A(x) = \pi_A \circ \theta_A^x(\mathrm{id}_A) = \theta_A^{\beta_A(x)} \circ \iota_A^x(\mathrm{id}_A) = \theta_A^{\beta_A(x)}(\mathrm{id}_A) = \xi'_A \circ \beta_A(x).$$

Thus (ξ, ξ') gives a morphism from $\beta\colon \mathrm{F} \to \mathrm{F}'$ to $\pi\colon \mathrm{G} \to \mathrm{G}'$.

By the proof of Theorem 1.2.4, the colimit of the diagram † is $\beta'\colon \mathrm{F} \to \mathrm{F}_\beta$ with F_β the functor taking A to the image of β_A and $f\colon A \to A'$ to $(\mathrm{F}'f)|_{\mathrm{im}(\beta_A)}$. In this case β'_A is β_A with the codomain changed to $\mathrm{im}(\beta_A)$.

In order to produce $\mathrm{F} \xrightarrow{\beta} \mathrm{F}'$ some amendments to † must be made. Construct the diagram ‡ by adding

1. a vertex $\mathrm{Hom}_{\mathsf{A}}(A, \_)'_y$ for each $y \in \mathrm{F}'(A) \setminus \mathrm{im}(\beta_A)$ for each object A in A
2. an arrow $\zeta'^{f,y}\colon \mathrm{Hom}_{\mathsf{A}}(A', \_)'_{(\mathrm{F}f)(y)} \to \mathrm{Hom}_{\mathsf{A}}(A, \_)'_y$ for each pair (f, y) with $f\colon A \to A'$ a morphism in A and $y \in \mathrm{F}'(A) \setminus \mathrm{im}(\beta_A)$

to the diagram †. The interpretation of the new data is that $\mathrm{Hom}_{\mathsf{A}}(A', \_)'_y$ maps to the zero natural transformation

$$\mathrm{Hom}_{\mathsf{A}}(A, \_)' \xrightarrow{0} \mathrm{Hom}_{\mathsf{A}}(A, \_)_y$$

while $\zeta'^{f,y}\colon \mathrm{Hom}_{\mathsf{A}}(A', \_)'_{(\mathrm{F}f)(y)} \to \mathrm{Hom}_{\mathsf{A}}(A, \_)_y$ maps to the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{A}}(A', _)' \arrow[r, "0"] \arrow[d, "0"] & \mathrm{Hom}_{\mathsf{A}}(A, _)' \arrow[d, "0"] \\
\mathrm{Hom}_{\mathsf{A}}(A', _)_{(\mathrm{F}'f)(y)} \arrow[r, "\zeta^{f,y}"] & \mathrm{Hom}_{\mathsf{A}}(A, _)_y.
\end{tikzcd}$$

From Lemma 1.2.6 and the proof of Theorem 1.2.4 it readily follows that the colimit of ‡ is $\beta\colon \mathrm{F} \to \mathrm{F}'$. □

Remark 1.2.7. Let Set_* be the category of pointed sets.

1. For any category C there is a result similar to Theorem 1.2.4 concerning $[\mathsf{C}, \mathsf{Set}]$ and $[\mathsf{C}, \mathsf{Set}_*]$.
2. For any category C there is a result similar to Proposition 1.2.5 concerning $[\mathsf{C}, \mathsf{Set}_*]$.

Moreover, the similar results concerning the covariant Yoneda embedding are also true, and the proofs are similar. ■

Recall that for an Ab-enriched category A the functor category $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ is additive (it is even abelian). It follows that there is a category of chain complexes over $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$, and Proposition 1.2.8 shows that this category can be identified with the category of Ab-functors from A to $\mathsf{Ch}(\mathsf{Ab})$.

Proposition 1.2.8. *Define the functor* $\Psi\colon \mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}) \to [\mathsf{A},\mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ *by*

$$\big(\Psi(\mathrm{F}_n,\delta_n)\big)(A) = \big(F_n(A),(\delta_n)_A\big)$$

on objects (F_n,δ_n) *and*

$$\big(\Psi(\eta_n)\big)_A = \big((\eta_n)_A\big)$$

on morphisms $(\eta_n)\colon (\mathrm{F}_n,\delta_n) \to (\mathrm{G}_n,\epsilon_n)$ *(where A is an object of* A*). Then Ψ is an isomorphism of categories, and the chain map (η_n) is null-homotopic if and only if $\Psi\big((\eta_n)\big)$ is null-homotopic.*

Proof. Let (F_n,δ_n) be an object of $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ and $f\colon A \to A'$ be a morphism in A. Since $(\mathrm{F}_{n-1}f)\circ(\delta_n)_A = (\delta_n)_{A'}\circ(\mathrm{F}_nf)$ by the naturality of δ_n, $\big(\mathrm{F}_n(f)\big)$ is a chain map $\big(\mathrm{F}_n(A),(\delta_n)_A\big) \to \big(\mathrm{F}_n(A'),(\delta_n)_{A'}\big)$. By looking at each degree of the chain complex it follows that $\big(\Psi(\mathrm{F}_n,\delta_n)\big)$ is an Ab-functor from A to $\mathsf{Ch}(\mathsf{Ab})$. For a morphism $(\eta_n)\colon (\mathrm{F}_n,\delta_n) \to (\mathrm{G}_n,\epsilon_n)$ the diagram

$$\begin{tikzcd}
\mathrm{F}_n(A) \arrow[r, "(\eta_n)_A"] \arrow[d, "\mathrm{F}_nf"] & \mathrm{G}_n(A) \arrow[d, "\mathrm{G}_nf"] \\
\mathrm{F}_n(A') \arrow[r, "(\eta_n)_{A'}"] & \mathrm{G}_n(A')
\end{tikzcd}$$

commutes since $\eta_n\colon \mathrm{F}_n \to \mathrm{G}_n$ is a natural transformation. Thus $\Psi(\eta_n)$ gives a natural transformation $\big(\Psi(\mathrm{F}_n,\delta_n)\big) \to \big(\Psi(\mathrm{G}_n,\epsilon_n)\big)$.

An inverse to Ψ is given by $\Phi\colon [\mathsf{A},\mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}} \to \mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$, where the functor $\mathrm{H}\colon A \mapsto \big((\mathrm{H}A)_n, d_n^A\big)$ is mapped to (H_n,δ_n) with $\mathrm{H}_n(A) = (\mathrm{H}A)_n$ and $(\delta_n)_A = d_n^A$ for objects A in A. For natural transformations $\theta\colon \mathrm{H} \to \mathrm{K}$, the functor Φ gives the chain map (θ_n), where $(\theta_n)_A = (\theta_A)_n$.

It is clear that if (η_n) is null-homotopic, so is $\big((\eta_n)_A\big)$ for all objects A in A. Now assume that $\Psi\big((\eta_n)\big)$ is null-homotopic. Then for each A there is a natural degree one map $(s_A)_n\colon \mathrm{F}_nA \to \mathrm{G}_{n+1}A$ such that $(\eta_n)_A = (s_A)_{n-1}\circ(\delta_n)_A + (\epsilon_{n+1})_A\circ(s_A)_n$. The maps s_A then give the desired degree one map $(\mathrm{F}_n,\delta_n) \to (\mathrm{G}_n,\epsilon_n)$. $\square$

As noted in Section 1.1, the category $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ has both a suspension functor and mapping cones. From Proposition 1.2.8 there is the following alternate description of mapping cones:

Corollary 1.2.9. *Given a morphism $\{\eta_n\}\colon \{\mathrm{F}_n\} \to \{\mathrm{G}_n\}$ in $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$, let C^η be the mapping cone of $\{\eta_n\}$. Then for each A in* A *the identity map gives a natural isomorphism $C^\eta(A) \simeq \mathrm{cone}(\{\eta_n\}_A)$ where $\mathrm{cone}(\{\eta_n\}_A)$ is the mapping cone of the chain map $\{(\eta_n)_A\}\colon \{\mathrm{F}_n(A)\} \to \{\mathrm{G}_n(A)\}$.* $\square$

The total tensor product on $\mathsf{Ch}(\mathsf{Ab})$ gives rise to an "external" tensor product in the category $[\mathsf{A},\mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$:

Definition 1.2.10. If F, G are in $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$, then their *tensor product*, $\mathrm{F} \otimes_T \mathrm{G}$, is the functor that takes an object A in A to $\mathrm{F}(A) \otimes_T \mathrm{G}(A)$, and a morphism $f \colon A \to B$ in A is taken to

$$\mathrm{F}f \otimes \mathrm{G}f \colon \mathrm{F}(A) \otimes_T \mathrm{G}(A) \to \mathrm{F}(B) \otimes_T \mathrm{G}(B)$$

with components

$$\begin{aligned}(\mathrm{F}f \otimes \mathrm{G}f)_n \colon \bigoplus_{i+j=n} \mathrm{F}(A)_i \otimes \mathrm{G}(A)_j &\to \bigoplus_{i+j=n} \mathrm{F}(B)_i \otimes \mathrm{B}(B)_j \\ a'_i \otimes a''_j &\mapsto (\mathrm{F}f)_i(a'_i) \otimes (\mathrm{G}f)_j(a''_j).\end{aligned}$$ ♠

As mentioned earlier, if the category A have a tensor product $\otimes_\sigma$, then Day constructed a internal tensor product in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ (Definition 1.2.2). By the same technique, the tensor product on $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ can be internalised.

Definition 1.2.11. The *internal tensor product* of $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ is given by taking the left Kan extension of the external tensor product along $\otimes_\sigma$, that is given two functors F_1, F_2 in $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ then their internal tensor product $\mathrm{F}_1 \underline{\otimes} \mathrm{F}_2$ is given by

$$(\mathrm{F}_1 \underline{\otimes} \mathrm{F}_2)(A) = \operatorname*{colim}_{A_1 \otimes_\sigma A_2 \to A} \mathrm{F}_1(A_1) \otimes_T \mathrm{F}_2(A_2).$$ ♠

By combining Definition 1.2.11, Definition 1.1.9 and Remark 1.1.10 one arrives at the definition of an internal hom object in $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$:

Definition 1.2.12. For functors G, H in $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ define the *internal hom object* as the functor $\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H}) \in [\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ given by

$$\big(\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A\big)_i = \mathrm{Nat}\left(\mathrm{O} \circ \mathrm{G}(\_), \mathrm{O} \circ \Sigma^{-i} \circ \mathrm{H}(A \otimes_\sigma \_)\right)$$

in degree i for an object A in A (O is the functor that forgets differentials from Remark 1.1.10). The differential is given in degree i by taking the natural transformation

$$\alpha \colon \mathrm{O} \circ \mathrm{G}(\_) \to \mathrm{O} \circ \Sigma^{-i} \circ \mathrm{H}(A \otimes_\sigma \_)$$

to the natural transformation

$$D_i\alpha \colon \mathrm{O} \circ \mathrm{G}(\_) \to \mathrm{O} \circ \Sigma^{-(i-1)} \circ \mathrm{H}(A \otimes_\sigma \_),$$

which for each B in A and $j \in \mathbf{Z}$ has components

$$\big((D_i\alpha)_B\big)_j = (-1)^{i-1}(\alpha_B)_{j-1} \circ d''_j + d^\pi_{i+j} \circ (\alpha_B)_j$$

where d'' is the differential on $\mathrm{G}(B)$ and d^π is the differential on $\mathrm{H}(A \otimes_\sigma B)$. For morphisms $f \colon A \to A'$ in A the functor is given in degree i by

$$\begin{aligned}\Big(\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})f\Big)_i \colon \Big(\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A'\Big)_i &\to \Big(\underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A\Big)_i \\ \eta &\mapsto \big(\mathrm{O} \circ \Sigma^{-i} \circ \mathrm{H}(f \otimes_\sigma \mathrm{id}\_)\big) \circ \eta.\end{aligned}$$ ♠

Proposition 1.2.13. *For* $\mathrm{F}, \mathrm{G}, \mathrm{H}$ *in* $[\mathsf{A}, \mathsf{Ch}(\mathsf{Ab})]_{\mathsf{Ab}}$ *there is a natural isomorphism*

$$\mathrm{Nat}\Big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})\Big) \to \mathrm{Nat}\Big(\mathrm{F}\underline{\otimes}\mathrm{G}, \mathrm{H}\Big).$$

The inverse to this isomorphism takes $\gamma\colon \big(\mathrm{F}\underline{\otimes}\mathrm{G}\big) \to \mathrm{H}$ *to* $\omega\colon \mathrm{F} \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})$ *given by*

$$\big(\omega^i_A(x)\big)^j_B(y) = \gamma^{i+j}_{A\otimes_\sigma B}(x \otimes y)$$

for $x \in (\mathrm{F}A)_i$ *and* $y \in (\mathrm{G}B)_j$.

Proof. This proof is rather lengthy and technical. The first part constructs the desired natural map, the second part shows that it is indeed a chain map, while the third part shows that it is an isomorphism.
Construction of the map: In order to specify a natural transformation $\gamma\colon \mathrm{F}\underline{\otimes}\mathrm{G} \to \mathrm{H}$, one must give the components $\gamma_A\colon (\mathrm{F}\underline{\otimes}\mathrm{G})A \to \mathrm{H}A$. Since $\underline{\otimes}$ is defined by a colimit, the components can equally well be given by a collection of compatible (in the sense of Definition 1.2.11) chain maps

$$\gamma_{A_1,A_2,f}\colon \mathrm{F}(A_1) \otimes_T \mathrm{G}(A_2) \to \mathrm{H}(A)$$

indexed over morphisms $f\colon A_1\otimes_\sigma A_2 \to A$ in A.

Now given a natural transformation $\beta\colon \mathrm{F} \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})$, there is for each object A_1 of A a natural (in A_1) map of chain complexes $\beta_{A_1}\colon \mathrm{F}A_1 \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})A_1$. So for each $i \in \mathbf{Z}$ one has a group homomorphism

$$\beta^i_{A_1}\colon (\mathrm{F}A_1)_i \to \mathrm{Nat}\left(\mathrm{O} \circ \mathrm{G}, \mathrm{O} \circ \Sigma^{-i} \circ \mathrm{H}(A_1\otimes_\sigma \_)\right)$$

commuting with the appropriate differentials. Thus for each $x \in (\mathrm{F}A_1)_i$, $A_2 \in \mathsf{A}$ and $j \in \mathbf{Z}$ there is a group homomorphism

$$\big(\beta^i_{A_1}(x)\big)^j_{A_2}\colon (\mathrm{G}A_2)_j \to \mathrm{H}(A_1\otimes_\sigma A_2)_{i+j}$$

that is natural in A_2. This gives rise to $\mathbf{Z}$-bilinear maps

$$\beta^{i,j,\times}_{A_1,A_2}\colon (\mathrm{F}A_1)_i \times (\mathrm{G}A_2)_j \to \big(\mathrm{H}(A_1\otimes_\sigma A_2)\big)_{i+j}, \quad (x, y) \mapsto \big(\beta^i_{A_1}(x)\big)^j_{A_2}(y)$$

so there are group homomorphisms

$$\beta^{i,j}_{A_1,A_2}\colon (\mathrm{F}A_1)_i \otimes (\mathrm{G}A_2)_j \to \big(\mathrm{H}(A_1\otimes_\sigma A_2)\big)_{i+j}, \quad x \otimes y \mapsto \big(\beta^i_{A_1}(x)\big)^j_{A_2}(y).$$

These group homomorphisms can be combined to a group homomorphism

$$\beta_{A_1,A_2}\colon \mathrm{F}(A_1) \otimes_T \mathrm{G}(A_2) \to \mathrm{H}(A_1\otimes_\sigma A_2).$$

In order to form a natural transformation $\gamma_\beta\colon \mathrm{F}\underline{\otimes}\mathrm{G} \to \mathrm{H}$ it is enough to look at morphisms $f\colon A_1\otimes_\sigma A_2 \to A$ in A. Given such a morphism f, consider the group homomorphism

$$\mathrm{H}(f) \circ \beta_{A_1,A_2}\colon \mathrm{F}(A_1) \otimes_T \mathrm{G}(A_2) \to \mathrm{H}(A).$$

In order to get a map from the colimit to $\mathrm{H}(A)$, such maps have to satisfy the compatibility requirement. Let

$$\begin{tikzcd} A_1\otimes_\sigma A_2 \arrow[rr, "h_1\otimes h_2"] \arrow[dr, "f"'] & & A'_1\otimes_\sigma A'_2 \arrow[dl, "f'"] \\ & A & \end{tikzcd}$$

be a commutative triangle in A and consider

$$\beta^{i,j}_{h_1\otimes h_2} = \beta^{i,j}_{A'_1,A'_2} \circ \big((\mathrm{F}h_1)_i \otimes (\mathrm{G}h_2)_j\big) \colon (\mathrm{F}A_1)_i \otimes (\mathrm{G}A_2)_j \to \big(\mathrm{H}(A'_1\otimes_\sigma A'_2)\big)_{i+j}.$$

Then

$$\begin{aligned}
\beta^{i,j}_{h_1\otimes h_2}(x\otimes y) &= \left(\Big(\big(\beta^i_{A'_1}\circ(\mathrm{F}h_1)_i\big)(x)\Big)^j_{A'_2} \circ \Big(\mathrm{G}h_2\Big)_j\right)(y)\\
&=^\dagger \left(\Big(\Sigma^{-i}\circ \mathrm{H}(A'_1\otimes_\sigma _)h_2\Big)_j \circ \Big(\big(\beta^i_{A'_1}\circ(\mathrm{F}h_1)_i\big)(x)\Big)^j_{A_2}\right)(y)\\
&= \left(\Big(\mathrm{H}(\mathrm{id}_{A'_1}\otimes_\sigma h_2)\Big)_{i+j} \circ \Big(\big(\beta^i_{A'_1}\circ(\mathrm{F}h_1)_i\big)(x)\Big)^j_{A_2}\right)(y)\\
&=^\ddagger \left(\Big(\mathrm{H}(\mathrm{id}_{A'_1}\otimes_\sigma h_2)\Big)_{i+j} \circ \left(\Big(\big(\underline{\mathrm{Hom}}(\mathrm{G},\mathrm{H})h_1\big)_i \circ \beta^i_{A_1}\Big)(x)\right)^j_{A_2}\right)(y)\\
&= \left(\Big(\mathrm{H}(\mathrm{id}_{A'_1}\otimes_\sigma h_2)\Big)_{i+j} \circ \left(\Big(\Sigma^{-i}\circ \mathrm{H}(h_1\otimes_\sigma \mathrm{id}_)\circ \beta^i_{A_1}\Big)(x)\right)^j_{A_2}\right)(y)\\
&= \left(\Big(\mathrm{H}(\mathrm{id}_{A'_1}\otimes_\sigma h_2)\Big)_{i+j} \circ \Big(\mathrm{H}(h_1\otimes_\sigma \mathrm{id}_{A_2})\Big)_{i+j} \circ \Big(\beta^i_{A_1}(x)\Big)^j_{A_2}\right)(y)
\end{aligned}$$

where $\dagger$ holds since $\beta^k_{A'_1}\big((\mathrm{F}h_1)_k(x)\big)$ is a natural transformation from $\mathrm{O}\circ\mathrm{G}$ to $\mathrm{O}\circ\Sigma^{-k}\circ \mathrm{H}(A'_1\otimes_\sigma\_)$, and $\ddagger$ is true because β is a natural transformation from F to $\underline{\mathrm{Hom}}(\mathrm{G},\mathrm{H})$. This shows that the group homomorphisms

$$\gamma_{A_1,A_2,f} = \mathrm{H}(f)\circ\beta_{A_1,A_2}\colon \mathrm{F}(A_1)\otimes_T \mathrm{G}(A_2) \to \mathrm{H}(A)$$

are compatible in the sense of Definition 1.2.11, and so give rise to a natural map

$$\gamma_\beta\colon \mathrm{O}\circ\big(\mathrm{F}\underline{\otimes}\mathrm{G}\big) \to \mathrm{O}\circ\mathrm{H}.$$

By a careful examination of the steps in the construction of γ, it is clear that the map $\mathrm{Nat}\Big(\mathrm{F},\underline{\mathrm{Hom}}(\mathrm{G},\mathrm{H})\Big) \to \mathrm{Nat}\Big(\mathrm{F}\underline{\otimes}\mathrm{G},\mathrm{H}\Big)$ is natural in F, G and H.

The map is a chain map: In order to show that $\gamma_\beta\colon \big(\mathrm{F}\underline{\otimes}\mathrm{G}\big)\to\mathrm{H}$ is a chain map it is enough to show that the maps β_{A_1,A_2} are chain maps. So consider $x\otimes y\in(\mathrm{F}A_1)_i\otimes(\mathrm{G}A_2)_j$ and let d be the differential on $\mathrm{F}(A_1)\otimes_T\mathrm{G}(A_2)$, d' the differential on $\mathrm{F}(A_1)$, d'' the differential on $\mathrm{G}(A_2)$, and D the differential on $\underline{\mathrm{Hom}}(\mathrm{G},\mathrm{H})(A_1)$. Then

$$\begin{aligned}
(\beta_{A_1,A_2})_{i+j-1}\circ d_{i+j}(x\otimes y) &= (\beta_{A_1,A_2})_{i+j-1}\big(d'_i(x)\otimes y + (-1)^i x\otimes d''_j(y)\big)\\
&= \big(\beta^{i-1}_{A_1}(d'_i(x))\big)^j_{A_2}(y) + (-1)^i\big(\beta^i_{A_1}(x)\big)^{j-1}_{A_2}\big(d''_j(y)\big)\\
&= \big(D_i\circ\beta^i_{A_1}(x)\big)^j_{A_2}(y) + (-1)^i\big(\beta^i_{A_1}(x)\big)^{j-1}_{A_2}\big(d''_j(y)\big).
\end{aligned}$$

Now note that if d^π is the differential on $\mathrm{H}(A_1 \otimes_\sigma A_2)$ then

$$\big(D_i \circ \beta^i_{A_1}(x)\big)^j_{A_2}(y) = (-1)^{i-1}\big(\beta^i_{A_1}(x)\big)^{j-1}_{A_2} \circ d''_j(y) + d^\pi_{i+j} \circ \big(\beta^i_{A_1}(x)\big)^j_{A_2}(y),$$

whence β_{A_1,A_2} is a chain map.
The map is an isomorphism: For injectivity, if $\alpha, \beta \colon \mathrm{F} \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})$ are distinct, then there is an $A_1 \in \mathsf{A}$, $i \in \mathbf{Z}$ and $x \in (\mathrm{F}A_1)_i$ such that $\alpha^i_{A_1}(x) \neq \beta^i_{A_1}(x)$. Thus there exists $A_2 \in \mathsf{A}$, $j \in \mathbf{Z}$ and some $y \in (\mathrm{G}A_2)_j$ such that $\big(\alpha^i_{A_1}(x)\big)^j_{A_2}(y) \neq \big(\beta^i_{A_1}(x)\big)^j_{A_2}(y)$, whence $\gamma_\alpha \neq \gamma_\beta$.

For surjectivity, given $\gamma \colon \big(\mathrm{F} \underline{\otimes} \mathrm{G}\big) \to \mathrm{H}$ define $\omega \colon \mathrm{F} \to \underline{\mathrm{Hom}}(\mathrm{G}, \mathrm{H})$ by

$$\big(\omega^i_A(x)\big)^j_B(y) = \gamma^{i+j}_{A \otimes_\sigma B}(x \otimes y)$$

for $x \in (\mathrm{F}A)_i$ and $y \in (\mathrm{G}B)_j$. This is well-defined since γ is a natural transformation. Since the formula for $\omega^{i,j}_{A,B}(x \otimes y)$ is identical to that of $\gamma^{i+j}_{A \otimes_\sigma B}(x \otimes y)$ it follows that $\gamma = \gamma_\omega$. $\square$

Since the representable objects are dense in both $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ and its arrow category, there is hope that a similar density statement holds for $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$. This is indeed the case, as Proposition 1.2.14 shows.

Proposition 1.2.14. *The objects of* $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ *(i.e. chain complexes of* Ab*-functors* A *to* Ab*) are colimits of chain complexes of representable functors.*

Proof. Assume (F_n, δ_n) is an object of $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$. Define the object $(\mathrm{F}^k_n, \delta^k_n)$ by

$$\mathrm{F}^k_n = \begin{cases} \mathrm{F}_n & n \le k \\ 0 & n > k, \end{cases} \quad \text{and } \delta^k_n = \begin{cases} \delta_n & n \le k \\ 0 & n > k. \end{cases}$$

and note that there are inclusions $(\mathrm{F}^k_n, \delta^k_n) \hookrightarrow (\mathrm{F}^{k+1}_n, \delta^{k+1}_n)$. Moreover, it is clear that $\mathrm{colim}_k(\mathrm{F}^k_n, \delta^k_n) = (\mathrm{F}_n, \delta_n)$. The result now follows if $(\mathrm{F}^k_n, \delta^k_n)$ is the colimit of chain complexes of representable functors, and this will be done using transfinite induction and Proposition 1.2.5. Thus, for the remainder of the proof, assume (F_n, δ_n) is zero in degrees $n > k$.
Base case: The aim is to construct a diagram † such that the colimit of the diagram is F_k in degree k, $\mathrm{im}\,\delta_k$ in degree $k-1$ and 0 otherwise. This can be done as follows: Given $A \in \mathsf{A}$ and $x \in \mathrm{F}_k(A)$ let the diagram † have a vertex $V^{A,k,x}$. Moreover, for each morphism $f \colon A \to A'$ in A let † have an arrow $E^{f,k,x} \colon V^{A',k,(\mathrm{F}_k f)(x)} \to V^{A,k,x}$. View † as a diagram in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ in the following way: The vertex $V^{A,k,x}$ is the chain complex (V_n, d_n) with

$$V_n = \begin{cases} \mathrm{Hom}_{\mathsf{A}}(A, \_)_x & n = k, \\ \mathrm{Hom}_{\mathsf{A}}(A, \_)_{(\delta_k)_A(x)} & n = k-1, \\ 0 & \text{otherwise,} \end{cases} \text{ and } d_n = \begin{cases} \mathrm{id}_{\mathrm{Hom}_{\mathsf{A}}(A,\_)} & n = k, \\ 0 & n \neq k \end{cases}$$

while the arrow $E^{f,k,x}$ is the chain map (ζ_n) with

$$\zeta_n = \begin{cases} f^* & n = k, n = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

From the proof of Proposition 1.2.5 it is clear that the colimit of this diagram is a functor F' that is F_k in degree k, $\operatorname{im} \delta_k$ in degree $k-1$ and 0 otherwise.

Inductive step, successor ordinal: Assume there is a diagram † of representative functors such that $\operatorname{colim} \dagger = (\mathrm{F}^l_n, \delta^l_n)$ where $\delta^l_n \simeq \delta_n$ for $n > l$, $\delta^l_n = 0$ for $n < l$ and $\delta^l_l \colon \mathrm{F}_l \to \mathrm{G}$ where G is the functor that sends A to the image of $(\delta_l)_A$ in $\mathrm{F}_{l-1}(A)$.

The extension of † to the diagram ‡ is done as follows: For $A \in \mathsf{A}$ and $x \in \mathrm{F}_{l-1}(A) \setminus \operatorname{im}(\delta_l)_A$ add a new vertex $V^{A,l-1,x}$, and for each morphism $f \colon A \to A'$ in A add an arrow $E^{f,l-1,x} \colon V^{A',l-1,(\mathrm{F}_{l-1}f)(x)} \to V^{A,l-1,x}$.

Using Lemma 1.2.6 it is clear that the extension of † to ‡ has no effect on the colimit in degrees greater than l, and another application of the proof of Proposition 1.2.5 shows that the colimit of ‡ has the desired properties.

Inductive step, limit ordinal: There is only one limit ordinal to consider, namely the first infinite ordinal ω. Since the result holds for all finite ordinals, for each $m \in \mathbf{N}$ there is a representable chain complex C^m and a chain map $\mathrm{C}^m \to (\mathrm{F}_n, \delta_n)$ which is an isomorphism in degrees above $k - (m+1)$ and zero in degrees below $k - (m+1)$. It follows that there are chain maps $\mathrm{C}^m \to \mathrm{C}^{m+1}$. Let C^ω be the colimit of these chain maps, and note that C^ω is isomorphic to (F_n, δ_n).

Conclusion: By transfinite induction it follows that (F_n, δ_n) is the colimit of representable objects, so the result follows. A similar procedure can also be used to show that the objects of the arrow category of $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ are colimit of representable objects, cf. the proof of Proposition 1.2.5. □

1.3 Triangulated structure

The axioms of a triangulated categories was formulated by Jean-Louis Verdier in 1963. He was working with Grothendieck, and their aim was to develop a tool for studying cohomology theories in algebraic geometry. Of particular interest was the derived functor Ext, which Verdier studied within the framework of derived categories in [Ver77].

At the same time, in algebraic topology, Dieter Puppe built sequences out of cones and suspension (i.e. the cofibration or Puppe sequence) in the paper [Pup58]. This lead to him formulating axioms similar to the ones for a triangulated category in [Pup62].

There are several equivalent definitions of a triangulated category. The one below is from [May01].

Definition 1.3.1. Let T be an additive category and Σ an additive equivalence $\mathsf{T} \to \mathsf{T}$ (called suspension, shift or translation). A *triangle* (f, g, h) in T is a diagram on the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X,$$

while a morphism of triangles $(f,g,h) \to (f',g',h')$ is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[r, "g"] \arrow[d, "j"] & Z \arrow[r, "h"] \arrow[d, "k"] & \Sigma X \arrow[d, "\Sigma i"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X.
\end{tikzcd}$$

The category T is a *triangulated category* if it has a collection of triangles (called distinguished triangles) satisfying the following axioms:

T1 For any object X and morphism f in T the following holds:

1. The triangle $X \xrightarrow{\mathrm{id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished.
2. The morphism f is part of a distinguished triangle (f, ι_f, π_f).
3. Any triangle isomorphic to a distinguished triangle is distinguished.

T2 If (f,g,h) is distinguished, so is $(g,h,-\Sigma f)$.

T3 Given distinguished triangles (f,f',f''), (g,g',g''), and (h,h',h'') such that $h = g \circ f$ then there exist morphisms j, j' such that (j,j',j'') is a distinguished triangle with $j'' = \Sigma f' \circ g''$ and the diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "h"] \arrow[dr, "f"] & & Z \arrow[rr, bend left, "g'"] \arrow[dr, "h'"] & & W \arrow[rr, bend left, "j''"] \arrow[dr, "g''"] & & \Sigma U \\
& Y \arrow[ur, "g"] \arrow[dr, "f'"] & & V \arrow[ur, "j'"] \arrow[dr, "h''"] & & \Sigma Y \arrow[ur, "\Sigma f'"] & \\
& & U \arrow[ur, "j"] \arrow[rr, bend right, "f''"] & & \Sigma X \arrow[ur, "\Sigma f"] & &
\end{tikzcd}$$

commutes. Such a diagram is called a *braid of distinguished triangles* generated by $h = g \circ f$ or cogenerated by $j'' = \Sigma f' \circ g''$. ♠

In [Ver77] the axioms for a triangulated category are denoted by TR1, TR2, TR3, and TR4. The axioms TR1 and T1 are identical, while TR4 (the octahedron axiom) is a different formulation of T3. Moreover, both TR3 and TR2 are consequences of the axioms above [May01].

Another set of axioms, denoted TR0, TR1, TR2, TR3, and TR4', is given in [Nee01]. In this case the combination of TR0 and TR1 is equal to T1, while TR2 and TR3 are identical to their namesake in [Ver77]. The last axiom, TR4', is a stronger variant of TR3, and it can be shown that TR4 and TR4' are equivalent in the presence of the axioms TR0, TR1, TR2, and TR3 [Nee91].

Example 1.3.2. Let C be a pointed category. If there is a model category structure (see Section 1.4) on C, then $\mathsf{Ho}(\mathsf{C})$ obtain both a suspension and a loop space functor [Qui67, Section I.2, Theorem 2]. The model category structure is *stable* if the loop space functor is an inverse equivalence to the suspension functor. For such a *stable model category* its homotopy category is triangulated [Hov99, Sections 6.1 – 6.5 and Proposition 7.1.6]. ♣

One of the first instances of a stable homotopy category in algebraic topology was the Spanier–Whithead category, which was introduced in [SW53]. In the original category, the objects are pointed CW-complexes, and a morphism $f\colon X \to Y$ is an element of $\operatorname{colim}_{k>=0} \operatorname{Hom}_{\mathsf{Top}_*}(\mathbf{S}^k \wedge X, \mathbf{S}^k \wedge Y)$. In order to obtain a triangulated category one must invert the suspension functor, so the modern version of the Spanier–Whithead has objects pairs (X, n) where X is a CW-complex and n is an integer, while a morphism $(X, n) \to (Y, m)$ is an element of $\operatorname{colim}_{k>=0} \operatorname{Hom}_{\mathsf{Top}_*}(\mathbf{S}^{n+k} \wedge X, \mathbf{S}^{m+k} \wedge Y)$. By [Mar83, p. 8, Theorem 7] this is a triangulated category where the distinguished triangles are isomorphic to

$$(X, l) \xrightarrow{(f, \mathrm{id}_l)} (Y, l) \longrightarrow (\mathrm{cone}\,(f), l) \longrightarrow (\Sigma X, l).$$

There is a similar category in the setting of non-commutative geometry, namely the category $\Sigma\mathrm{Ho}$ of [CMR07]. Moreover, this category can be used as building block for the category KK.

Example 1.3.3. In the theory of operator algebras, it is known that KK-theory introduced by Kasparov in [Kas80] yields an abelian category KK, and by [MN06, Appendix A] KK is also triangulated. In this category, the distinguished triangles are of the form

$$\mathrm{S}B \longrightarrow \mathrm{C}_f \longrightarrow A \xrightarrow{f} B$$

where $\mathrm{S}B = B \otimes_\sigma \mathrm{C}_0(\mathbf{R}) \simeq B \otimes_\sigma \mathrm{C}_0\big((0,1)\big) \simeq \mathrm{C}_0\big((0,1), B\big)$ is the C*-algebra suspension of B and $\mathrm{C}_f = \Big\{(\beta, a) \in \mathrm{C}_0\big((0,1], B\big) \oplus A \;\Big|\; f(a) = \beta(1)\Big\}$ is the C*-algebra mapping cone of f. Moreover, due to Bott periodicity the suspension endofunctor is its own inverse in KK [Bla98, Corollary 19.2.2]. ♣

Recall that the homotopy category of $\mathsf{Ch}(\mathsf{C})$, $K(\mathsf{C})$, has the same objects as $\mathsf{Ch}(\mathsf{C})$, while the morphisms are equivalence classes of chain maps, with $f \sim g$ if and only if $f - g$ is null-homotopic.

Example 1.3.4. Consider the category $\mathsf{Ch}(\mathsf{Ab})$ of chain complexes of abelian groups. Let f, g be chain maps, $h = g \circ f$, and consider the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "h"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left, "\iota_g"] \arrow[dr, "\iota_h"] & & \mathrm{cone}\,(g) \arrow[rr, bend left, "\Sigma\iota_f \circ \pi_g"] \arrow[dr, "\pi_g"] & & \Sigma\mathrm{cone}\,(f) \\
& Y \arrow[ur, "g"'] \arrow[dr, "\iota_f"'] & & \mathrm{cone}\,(h) \arrow[ur, "j'"'] \arrow[dr, "\pi_h"] & & \Sigma Y \arrow[ur, "\Sigma\iota_f"'] & \\
& & \mathrm{cone}\,(f) \arrow[ur, "j"] \arrow[rr, bend right, "\pi_f"'] & & \Sigma X \arrow[ur, "\Sigma f"'] & &
\end{tikzcd}$$

where

$$j_n\colon \mathrm{cone}\,(f)_n \to \mathrm{cone}\,(h)_n, \quad (x_1, y) \mapsto \big(x_1, g_n(y)\big)$$

and

$$j_n'\colon \operatorname{cone}(h)_n \to \operatorname{cone}(g)_n, \quad (x_1, z) \mapsto \big(f_{n-1}(x_1), z\big).$$

Since $\operatorname{cone}(g)$ need not be isomorphic to $\operatorname{cone}(j)$ (i.e. if the complex X is different from the zero complex), it is clear that the collection

$$\Big\{(f, \iota_f, \pi_f) \;\Big|\; f \text{ is a chain map}\Big\} \cup \Big\{(\mathrm{id}_X, 0, 0) \;\Big|\; X \text{ is a chain complex}\Big\}$$

does not give distinguished triangles in the category $\mathsf{Ch}(\mathsf{Ab})$ of chain complexes of abelian groups.

However, by working in the homotopy category, $K(\mathsf{Ab})$, of $\mathsf{Ch}(\mathsf{Ab})$ and defining $\tilde{f}$ to be the equivalence class of f, then $\Big\{(\tilde{f}, \tilde{\iota_f}, \tilde{\pi_f}) \;\Big|\; f \text{ is a chain map}\Big\}$ does give a triangulation*:

T1 The complex $\operatorname{cone}(\mathrm{id}_X)$ is contractible.

T2 Since the chain map

$$\begin{aligned}\operatorname{cone}(\iota_f) \to \operatorname{cone}(\iota_f)\colon Y_{n-1} \oplus X_{n-1} \oplus Y_n &\to Y_{n-1} \oplus X_{n-1} \oplus Y_n,\\ (y_1, x_1, y) &\mapsto -\big(y_1 + f_{n-1}(x_1), 0, y\big)\end{aligned}$$

is null-homotopic, the chain map

$$\begin{aligned}\Sigma X \to \operatorname{cone}(\iota_f)\colon X_{n-1} &\to Y_{n-1} \oplus X_{n-1} \oplus Y,\\ x_1 &\mapsto \big(-f_{n-1}(x_1), x_1, 0\big)\end{aligned}$$

induces the inverse to the projection

$$\begin{aligned}\operatorname{cone}(\iota_f) \to \Sigma X\colon Y_{n-1} \oplus X_{n-1} \oplus Y_n &\to X_{n-1},\\ (y_1, x_1, y) &\mapsto x_1.\end{aligned}$$

T3 The chain map

$$\begin{aligned}\operatorname{cone}(j) \to \operatorname{cone}(j)\colon X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n &\to X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n,\\ (x_2, y_1, x_1, z) &\mapsto \big(x_2, -f_{n-1}(x_1), x_1, 0\big)\end{aligned}$$

is also null-homotopic, so the chain map

$$\begin{aligned}\operatorname{cone}(j) \to \operatorname{cone}(g)\colon X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n &\to Y_{n-1} \oplus Z_n,\\ (x_2, y_1, x_1, z) &\mapsto \big(y_1 + f_{n-1}(x_1), z\big)\end{aligned}$$

induces the inverse to the inclusion

$$\begin{aligned}\operatorname{cone}(g) \to \operatorname{cone}(j)\colon Y_{n-1} \oplus Z_n &\to X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n,\\ (y_1, z) &\mapsto (0, y_1, 0, z).\end{aligned}$$

♣

* Note that the various chain maps described below are linear combinations of known chain maps. Thus Ab can be replaced with any abelian category.

Now form the homotopy category $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ of $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$. Since the homotopy category of ordinary chain complexes of abelian groups is triangulated (Example 1.3.4), it seems likely that $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ is triangulated with distinguished triangles sequences isomorphic to

$$\{\mathrm{F}_n\} \xrightarrow{\tilde{\eta}} \{\mathrm{G}_n\} \xrightarrow{\tilde{\iota_\eta}} \mathrm{cone}\,(\{\eta_n\}) \xrightarrow{\tilde{\pi_\eta}} \Sigma\{\mathrm{F}_n\}$$

for $\{\eta_n\}\colon \{\mathrm{F}_n\} \to \{\mathrm{G}_n\}$ in $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ and $\tilde{\zeta}$ the equivalence class of ζ in the homotopy category $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$. This is indeed the case:

Proposition 1.3.5. *The category $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ is triangulated with the above mentioned distinguished triangles. Moreover the isomorphism in Proposition 1.2.8 induces an isomorphism of categories $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}) \to [\mathsf{A},K(\mathsf{Ab})]_{\mathsf{Ab}}$.*

Proof. For the triangulated part, there are three axioms to check:

T1 Since the chain complex $\mathrm{cone}\,(\mathrm{id}_C)$ is contractible in a natural way for all chain complexes C, so is the functor $\mathrm{cone}\,(\mathrm{id}_\mathrm{F})$ for all functors $\mathrm{F}\colon \mathsf{A} \to \mathsf{Ch}(\mathsf{Ab})$. Moreover, any chain map of natural transformations η is the first map of a distinguished triangle.

T2 Assume $\eta\colon \mathrm{F} \to \mathrm{G}$ is a morphism in $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ and consider the commutative diagram

$$\begin{tikzcd}
\mathrm{G} \arrow[r, "\iota_\eta"] \arrow[d, "\mathrm{id}_\mathrm{G}"] & \mathrm{cone}\,(\eta) \arrow[r, "\iota_{\iota_\eta}"] \arrow[d, "\mathrm{id}_{\mathrm{cone}\,(\eta)}"] & \mathrm{cone}\,(\iota_\eta) \arrow[r, "\pi_{\iota_\eta}"] \arrow[d, "\phi"] & \Sigma\mathrm{G} \arrow[d, "\mathrm{id}_{\Sigma\mathrm{G}}"] \\
\mathrm{G} \arrow[r, "\iota_\eta"] & \mathrm{cone}\,(\eta) \arrow[r, "\pi_\eta"] & \Sigma\mathrm{F} \arrow[r, "-\Sigma\eta"] & \Sigma\mathrm{G}
\end{tikzcd}$$

where ϕ the projection onto $\Sigma\mathrm{F}$. By evaluating the functors on A, one obtain a corresponding diagram in $\mathsf{Ch}(\mathsf{Ab})$. By passing to the homotopy category the map $\tilde{\phi}_A$ has a natural inverse (see Example 1.3.4). This shows that $\tilde{\phi}$ is an isomorphism in $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$.

T3 This argument is similar to the one in T2: Evaluate the functors, and look at the corresponding diagram in $K(\mathsf{Ab})$ (cf. Example 1.3.4 for details).

The statement regarding the isomorphism of categories follows by the null-homotopic part of Proposition 1.2.8. ☐

The proof of Proposition 1.3.5 demonstrates that for any object $A \in \mathsf{A}$ the functor $K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}) \to K(\mathsf{Ab})$ given by evaluation on A preserves distinguished triangles. Functors with this property are called triangulated:

Definition 1.3.6. Let T_1 and T_2 be triangulated categories with suspensions Σ_1 and Σ_2 respectively. An additive functor $\mathrm{F}\colon \mathsf{T}_1 \to \mathsf{T}_2$ is *triangulated* if there exists a natural transformation $\eta\colon \mathrm{F}\circ\Sigma_1 \to \Sigma_2 \circ \mathrm{F}$ such that if

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma_1 X$$

is a distinguished triangle, so is

$$\mathrm{F}X \xrightarrow{\mathrm{F}f} \mathrm{F}Y \xrightarrow{\mathrm{F}g} \mathrm{F}Z \xrightarrow{\eta_X \circ \mathrm{F}h} \Sigma_2(\mathrm{F}X). \qquad \spadesuit$$

There is another triangulated category closely related to $\mathsf{Ch}(\mathsf{Ab})$, namely the derived category $D(\mathsf{Ab})$. This category has the same objects as $\mathsf{Ch}(\mathsf{Ab})$, but the hom sets are quite different. In this category all quasi-isomorphisms, that is chain maps that induce an isomorphism on all homology groups, are inverted. More precisely, $D(\mathsf{Ab})$ is the localisation of $K(\mathsf{Ab})$ on the collection of quasi-isomorphisms [Wei95, Chapter 10].

Since the quasi-isomorphisms form a multiplicative system ([Wei95, Definition 10.3.4 and Proposition 10.4.1]), one can give a more explicit description of the morphisms of $D(\mathsf{Ab})$: Any morphism $X \to Y$ in $D(\mathsf{Ab})$ can be written on the form fq^{-1} where $q\colon X' \to X$ is a quasi-isomorphism and $f\colon X' \to Y$ is a chain map. Two morphisms $f_1q_1^{-1}$ and $f_2q_2^{-1}$ in $D(\mathsf{Ab})$ are equal if there exists a commutative diagram

$$\begin{tikzcd} & X_1' \arrow[dl, "q_1"'] \arrow[dr, "f_1"] & \\ X & X' \arrow[l, "q"'] \arrow[r, "f"] \arrow[u] \arrow[d] & Y \\ & X_2' \arrow[ul, "q_2"] \arrow[ur, "f_2"'] & \end{tikzcd}$$

in $\mathsf{Ch}(\mathsf{Ab})$ with q a quasi-isomorphism*.

The composition of $fq^{-1}\colon X \to Y$ and $gr^{-1}\colon Y \to Z$ in $D(\mathsf{Ab})$ is given by $gf'(qr')^{-1}$ where f' and r' comes from the Ore condition of a multiplicative system. This is a condition which given $f\colon X' \to Y$ and a quasi-isomorphism $r\colon Y' \to Y$ provides a chain complex W, a chain map f', and a quasi-isomorphism r' such that the diagram

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "r'"'] & Y' \arrow[d, "r"] \\ X' \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

If $fq^{-1}\colon X \to Y$, $gr^{-1}\colon Y \to Z$ and $hs^{-1}\colon Z \to \Sigma X$ are morphisms in $D(\mathsf{Ab})$, then they form a distinguished triangle if there is a commutative diagram

$$\begin{tikzcd} & & & X''' \arrow[dll, "p"'] \arrow[dr, "l"] & & & \\ & & & & Y'' \arrow[dl, "t"'] \arrow[dr, "k"] & & \\ & X' \arrow[dl, "q"'] \arrow[dr, "f"] & & Y' \arrow[dl, "r"'] \arrow[dr, "g"] & & Z' \arrow[dl, "s"'] \arrow[dr, "h"] & \\ X & & Y & & Z & & \Sigma X \end{tikzcd}$$

* Thus fq^{-1} is a morphism in $D(\mathsf{Ab})$.

in $\mathsf{Ch}(\mathsf{Ab})$ where p and t are quasi-isomorphisms while

$$X'' \xrightarrow{l} Y'' \xrightarrow{k} Z' \xrightarrow{h} \Sigma X$$

is a distinguished triangle in $K(\mathsf{Ab})$.

Since $D(\mathsf{Ab})$ is a localisation of $K(\mathsf{Ab})$ there is a functor $K(\mathsf{Ab}) \to D(\mathsf{Ab})$. This functor is the identity on objects, while a morphism $f\colon X \to Y$ is mapped to $X = X \xrightarrow{f} Y$. Note that since the zero chain map is a quasi-isomorphism from any contractible chain complex to the zero chain complex, this is well-defined. It also follows that the functor $K(\mathsf{Ab}) \to D(\mathsf{Ab})$ is triangulated.

One can form the derived category for any additive category. In particular one can form the category $D([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ for any Ab-enriched category A, and it can be triangulated by following the recipe for $D(\mathsf{Ab})$. Namely, a triangle

$$\mathrm{F} \xrightarrow{\alpha\rho^{-1}} \mathrm{G} \xrightarrow{\beta\sigma^{-1}} \mathrm{H} \xrightarrow{\gamma\tau^{-1}} \Sigma\mathrm{F}$$

in $D([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ is triangulated if and only if there is a commutative diagram

$$\begin{tikzcd}
 & & & \mathrm{F}''' \arrow[ddll, "\upsilon"'] \arrow[dr, "\zeta"] & & & \\
 & & & & \mathrm{G}'' \arrow[dl, "\phi"'] \arrow[dr, "\eta"] & & \\
 & \mathrm{F}' \arrow[dl, "\rho"'] \arrow[dr, "\alpha"] & & \mathrm{G}' \arrow[dl, "\sigma"'] \arrow[dr, "\beta"] & & \mathrm{H}' \arrow[dl, "\tau"'] \arrow[dr, "\gamma"] & \\
\mathrm{F} & & \mathrm{G} & & \mathrm{H} & & \Sigma\mathrm{F}
\end{tikzcd}$$

in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ where υ and ϕ are quasi-isomorphisms while

$$\mathrm{F}''' \xrightarrow{\zeta} \mathrm{G}'' \xrightarrow{\eta} \mathrm{H}' \xrightarrow{\gamma} \Sigma\mathrm{F}$$

is a distinguished triangle in $K([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Note that if $\mathrm{F} = (\mathrm{F}_n, \delta_n)$ is in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$, then $(\ker \delta_n)(A) = \ker(\delta_n)_A$ and $(\operatorname{im} \delta_n)(A) = \operatorname{im}(\delta_n)_A$. Thus computing the nth homology group, $\mathrm{H}_n(\mathrm{F})$, gives a functor $\mathsf{A} \to \mathsf{Ab}$ such that $\big(\mathrm{H}_n(\mathrm{F})\big)(A) = \mathrm{H}_n\big(\mathrm{F}(A)\big)$. Consequentially, $\eta\colon \mathrm{F} \to \mathrm{G}$ is a quasi-isomorphism in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ if and only if $\eta_A\colon \mathrm{F}(A) \to \mathrm{G}(A)$ is a quasi-isomorphism in $\mathsf{Ch}(\mathsf{Ab})$ for all objects A in A. From this it follows that there is an isomorphism of categories $D([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}) \to [\mathsf{A}, D(\mathsf{Ab})]_{\mathsf{Ab}}$.

It can be hard to compute hom groups in $D(\mathsf{A})$, but in some cases this reduces to computations in $K(\mathsf{A})$:

Proposition 1.3.7 ([Wei95, Corollary 10.4.7])**.** *If P is a bounded below chain complex of projective objects, then* $\operatorname{Hom}_{D(\mathsf{A})}(P, X) \simeq \operatorname{Hom}_{K(\mathsf{A})}(P, X)$ *for all chain complexes X.*

The next lemma paves the way for using Proposition 1.3.7 in a special functor category setting.

Lemma 1.3.8. *If A is an object of A such that $\mathrm{Hom}_{\mathsf{A}}(A, A)$ is a free abelian group, then $\mathrm{Hom}_{\mathsf{A}}(A, \_)$ is projective in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$.*

Proof. Consider the diagram

$$\begin{tikzcd} & \mathrm{Hom}_{\mathsf{A}}(A, \_) \arrow[d, "\zeta"] & \\ \mathrm{F} \arrow[r, "\eta"] & \mathrm{G} \arrow[r] & 0 \end{tikzcd}$$

in $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$, and note that since $\mathrm{Hom}_{[\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}}(A, A)$ is a free abelian group, it is projective. Thus there is a homomorphism $\theta_A \colon \mathrm{Hom}_{[\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}}(A, A) \to \mathrm{F}(A)$ such that $\eta_A \circ \theta_A = \zeta_A$. For any object B in A define θ_B by $\theta_B(f) = \big(\mathrm{F}(f)\big)\big(\theta_A(\mathrm{id}_A)\big)$. This is clearly natural, and $\zeta_B(f) = \mathrm{G}(f) \circ \zeta_A(\mathrm{id}_A) = \mathrm{G}(f) \circ \eta_A \circ \theta_A(\mathrm{id}_A) = \eta_B \circ \theta_B(f)$. $\square$

1.4 Model category structure

The modern view of homotopy theory is through model categories, a concept introduced by Quillen in [Qui67]. Basically, the idea is that there might be several models for a "homotopy category". In each model there are morphisms (the weak equivalences) that becomes isomorphisms in the homotopy category. Sometimes, one would like to "improve" the model by adding more isomorphisms to its homotopy category. The usual technique used to do this is Bousfield localisation, which was introduced in [Bou75, appendix].

Recall that a model category is a category C with three classes of morphisms *weak equivalences* ($\xrightarrow{\simeq}$), *fibrations* ($\twoheadrightarrow$), and *cofibrations* ($\rightarrowtail$) such that each class contains the identity and is closed under composition. The category and classes are subject to the following five axioms:

MC1 Small limits (projective limits) and colimits (inductive limits) exist in C.

MC2 If f, g, and $g \circ f$ are morphisms in C, and two of them are weak equivalences, then so is the third.

MC3 Each class of morphisms ($\xrightarrow{\simeq}$, $\twoheadrightarrow$, and $\rightarrowtail$) is closed under retractions.

MC4 There exists a lift in the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "g"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

if i is an acyclic* cofibration and p is a fibration or if i is a cofibration and p is an acyclic* fibration.

MC5 Any morphism f in C factorizes functorially as $f = p_1 \circ i_1 = p_2 \circ i_2$ in where

- i_1 is a cofibration and p_1 is an acyclic fibration and
- i_2 is an acyclic cofibration and p_2 is a fibration.

One method for specifying model categories is by giving the weak equivalences and two collections I and J of morphisms. With some conditions on I and J this yields a *cofibrantly generated model category*, i.e. a model category "generated" by I and J. In such a model category the cofibrations and the fibrations are morphisms having lifting properties with respect to I and J. This ensures that MC1, MC2, MC3 and MC4 are given almost for free, while there is a device known as *the small object argument* (introduced by Quillen as [Qui67, Chapter II, p. 3.2, Lemma 3]) that gives MC5 (this requires some smallness conditions). The small object argument is in fact slightly stronger, as it shows that the class of cofibrations is the smallest class of morphisms containing I that is closed under retracts, transfinite composition and pushouts ([Hov99, Corollary 2.1.15]). An introduction to model categories (including the small object argument) can be found in [DS95].

Since liftings are central to such a specification, some more terminology is needed. First recall that if there is a lift $B \to X$ in all commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] & Y \end{tikzcd}$$

then i has the *left lifting property* with respect to p, and p the *right lifting property* with respect to i. Notice that by MC3 and MC5, if a morphism in a model category has the right lifting property with respect to all acyclic cofibrations then is must be a fibration. Thus the fibrations of a model category are precisely the morphisms that have the right lifting property with respect to all acyclic cofibrations. Correspondingly, the cofibrations are exactly the morphisms having the left lifting property with respect to all acyclic fibrations.

Definition 1.4.1. Let I be a class of morphisms in a category C.

- A morphism is in I-*proj* or is I-*projective* if it has the left lifting property with respect to all morphisms in I.
- A morphism is in I-*inj* or is I-*injective* if it has the right lifting property with respect to all morphisms in I.
- A morphism is in I-*fib* or is an I-*fibration* if it has the right lifting property with respect to every I-projective morphism.
- A morphism is in I-*cof* or is an I-*cofibration* if it has the left lifting property with respect to every I-injective morphism. ♠

* An acyclic fibration (cofibration) is a fibration (cofibration) that is also a weak equivalence.

Later in the text the following model category structure on chain complexes over a ring will be used. It is an example of a cofibrantly generated model category, the details can be found in [Hov99, Section 2.3].

Example 1.4.2. Let A be an abelian group and $n \in \mathbf{Z}$. The chain complex $\mathbf{S}^n(A)$ is A in degree n and 0 otherwise, while the chain complex $\mathbf{D}^n(A)$ is A in degrees n and $n-1$ (with the identity as the boundary map), and zero otherwise.

Let I be the collection of inclusions $\mathbf{S}^{n-1}(\mathbf{Z}) \to \mathbf{D}^n(\mathbf{Z})$ and J be the collection of chain maps $0 \to \mathbf{D}^n(\mathbf{Z})$. There is a cofibrantly generated model structure on $\mathsf{Ch}(\mathsf{Ab})$, the *projective model structure*, where the fibrations are the J-injectives, the cofibrations are the I-cofibrations, and weak equivalences are chain maps inducing an isomorphism on all homology groups.

In this model structure fibrations are levelwise surjections, so all objects are fibrant*. On the other hand, if a chain complex is cofibrant† then it must be projective in each degree. Using induction one can see that this is a sufficient criterion for chain complexes that are bounded below. However, [Hov99, Remark 2.4.7] shows that there exists a chain complex that is free in each degree but not cofibrant. ♣

As mentioned earlier, a model category is a model for a homotopy category. For any model category one can form the corresponding homotopy category:

Definition 1.4.3. Let C be a category and $\mathscr{W}$ a collection of morphisms in C called the weak equivalences. The *homotopy category* of C is denoted by $\mathsf{Ho}_{\mathscr{W}}(\mathsf{C})$ and is formed by inverting the weak equivalences. Formally the category $\mathsf{Ho}_{\mathscr{W}}(\mathsf{C})$ is $\mathsf{C}[\mathscr{W}^{-1}]$. If there is no ambiguity about the weak equivalences, $\mathsf{Ho}_{\mathscr{W}}(\mathsf{C})$ will be denoted $\mathsf{Ho}(\mathsf{C})$. ♠

A priori it is not clear that $\mathsf{Ho}(\mathsf{C})$ is a category. However, if C is a model category, then there is an alternate description of the hom sets in $\mathsf{Ho}(\mathsf{C})$. Namely $\mathrm{Hom}_{\mathsf{Ho}(\mathsf{C})}(X, Y) = \mathrm{Hom}_{\mathsf{C}}(\mathrm{Q}X, \mathrm{R}Y)/\sim$ where $\mathrm{Q}X$ is the cofibrant replacement of X, $\mathrm{R}Y$ is the fibrant replacement of Y, and $\sim$ is a specific equivalence relation [Hov99, Theorem 1.2.10 part ii]‡.

If one gives $\mathsf{Ch}(\mathsf{Ab})$ the projective model structure of Example 1.4.2, the weak equivalences are the quasi-isomorphisms. From this it is clear that $\mathsf{Ho}(\mathsf{Ch}(\mathsf{Ab}))$ is $D(\mathsf{Ab})$, the derived category of Ab.

Given a model structure on a category, one might want to increase the number of weak equivalences. An orderly method to do so is to use Bousfield localisations:

- Left Bousfield localisation: increase the number of weak equivalences and keep the number of cofibrations fixed, and consequentially decrease the class of fibrations.
- Right Bousfield localisation: increase the number of weak equivalences and keep the number of fibrations fixed, and consequentially decrease the class of cofibrations.

* An object is fibrant if the morphism from it to the terminal object, $*$, is a fibration. † An object is cofibrant if the morphism from the initial object, $\emptyset$, to it is a cofibration. ‡ So $\emptyset \rightarrowtail \mathrm{Q}X \xrightarrow{\simeq} X$ and $Y \xrightarrow{\simeq} \mathrm{R}Y \twoheadrightarrow *$.

Note that since there are more weak equivalences in the Bousfield localised model structure, there is an induced functor from the homotopy category of the original model category to the homotopy category of the Bousfield localised model category.

If the model category is "nice", there are theorems guaranteeing that there are Bousfield localisations. The first requirement is that of properness. Note that the pullback of a fibration is also a fibration, while a pushout of a cofibration is again a cofibration. Properness is a way to ensure that certain pushouts and pullbacks of a weak equivalence will again be a weak equivalence:

Definition 1.4.4. A model category is *left proper* if every pushout of a weak equivalence along a cofibration is a weak equivalence. Similarly, a model category is *right proper* if every pullback of a weak equivalence along a fibration is a weak equivalence. If a model category is both left and right proper, then it is *proper*. ♠

Example 1.4.5. The projective model category structure on $\mathsf{Ch}(\mathsf{Ab})$ is proper [CH02, Theorem 2.2]. ♣

The second requirement is that the model category should be either *combinatorial* or *cellular*. In both cases the model category has to be *cofibrantly generated**; in the combinatorial case the model category should also be *locally presented*†, while in the cellular case it is a bit more technical. The following theorem by Jeff Smith shows that Bousfield localisations exist in proper combinatorial model categories.

Theorem 1.4.6 ([Bar10, Theorem 4.7])**.** *If C is a left proper combinatorial model category and $\mathscr{H}$ is a set of morphisms in C, then the left Bousfield localisation $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ of C along $\mathscr{H}$ exists and satisfies:*

- *As categories, $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ and C are equal.*
- *As model category $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ is both left proper and combinatorial.*
- *The cofibrations of $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ and C are the same.*
- *The weak equivalences of $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ are the $\mathscr{H}$-local equivalences.*
- *The fibrant objects of $\mathsf{L}_{\mathscr{H}}\mathsf{C}$ are the fibrant $\mathscr{H}$-local objects of C.*

A similar theorem hold for left proper cellular model categories (the only modification is that the localised model structure is cellular instead of combinatorial) – this is [Hir03, Theorem 4.1.1].

In order to understand the above theorem, one need to understand the notion of $\mathscr{H}$-localness. This is a bit technical; a rough sketch for the case $\mathscr{H} = \{h\}$ for a morphism $h\colon C_1 \to C_2$ is presented below. If one inverts h in the homotopy category then there ought to be an isomorphism $h^*\colon \operatorname{Hom}_{\mathsf{Ho}(\mathsf{C})}(C_2, X) \to \operatorname{Hom}_{\mathsf{Ho}(\mathsf{C})}(C_1, X)$. Now, before localisation, there are several objects X such that the map h^* makes $\operatorname{Hom}(X, C_2)$ and $\operatorname{Hom}(X, C_1)$ "look alike", and such objects X are the $\{h\}$-local objects. Thus the $\{h\}$-local objects are the objects that in some sense thinks that h is invertible before localisation. It might happen that there is a morphism $h'\colon C_1' \to C_2'$ in C such that

* See Definition A.6. † See Definition A.8.

all the $\{h\}$-local objects thinks that $(h')^*$ makes $\mathrm{Hom}_{\mathsf{Ho(C)}}(C_2, X)$ and $\mathrm{Hom}_{\mathsf{Ho(C)}}(C_1, X)$ "look alike". Such morphism h' are $\{h\}$-local equivalences, and they are the morphisms that "makes things look alike" from the viewpoint of all $\{h\}$-local objects.

The "look alike" concept simplifies when $\mathsf{Ho(C)}$ is a triangulated category. In this case, if $h\colon C_1 \to C_2$ is invertible in the homotopy category then cone(h), the mapping cone of h, is isomorphic to 0 in the homotopy category. If the situation is nice enough, $\mathscr{H}$-localness can be described in terms of cone(h) for $h \in \mathscr{H}$. This idea was studied by Cisinski and Déglise in the setting of chain complexes, and by using Theorem 1.4.6 they showed that $\mathscr{H}$-localness reduces to questions in the derived category [CD09, Proposition 4.3].

The starting point of Cisinski and Déglise is a Grothendieck category (e.g. Ab or $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$) with a collection of objects $\mathcal{G}$. Using this they create the *$\mathcal{G}$-model structure* in [CD09, Theorem 2.5], whose construction is a generalisation of the one for Example 1.4.2.

Definition 1.4.7. Let C be a Grothendieck category and $\mathcal{G}$ a non-empty *set* of objects in C. The *$\mathcal{G}$-model structure* on $\mathsf{Ch(C)}$ has the quasi-isomorphisms as the weak equivalences, while the cofibrations is the smallest class of morphisms in $\mathsf{Ch(C)}$ closed under pushouts, transfinite composition, and retracts generated by morphisms $\mathbf{S}^{n-1}(\mathrm{G}) \to \mathbf{D}^n(\mathrm{G})$ for $\mathrm{G} \in \mathcal{G}$. ♠

Remark 1.4.8. A collection of morphisms that are closed under pushouts, transfinite composition, and retracts are called *weakly saturated*. From MC3 and the observation that the cofibrations are precisely the morphisms having the left lifting property with respect to all acyclic fibrations, it follows that the class of cofibrations in a model category is weakly saturated. ■

As mentioned earlier, instead of $\mathscr{H}$-localness for a collection of morphisms $\mathscr{H}$ the approach of Cisinski and Déglise uses the mapping cones of the morphisms in $\mathscr{H}$. Their approach is a bit more general, as they consider any set of objects $\mathscr{T}$.

Definition 1.4.9. Let C be a Grothendieck category and $\mathscr{T}$ be a set of objects in $\mathsf{Ch(C)}$. Assume $\mathsf{Ch(C)}$ has the $\mathcal{G}$-model structure.

- A chain complex K is *$\mathscr{T}$-flasque** if it is $\mathcal{G}$-fibrant and for any $T \in \mathscr{T}$ and $n \in \mathbf{Z}$ the group $\mathrm{Hom}_{D(\mathsf{C})}(T, \Sigma^n K)$ is trivial.
- A chain map $f\colon C_1 \to C_2$ is a *$\mathscr{T}$-local equivalence* if for any $\mathscr{T}$-flasque chain complex K the map $\mathrm{Hom}_{D(\mathsf{C})}(C_2, K) \to \mathrm{Hom}_{D(\mathsf{C})}(C_1, K)$ is a bijection.
- A chain complex K is *$\mathscr{T}$-local* if for any $\mathscr{T}$-local equivalence $f\colon C_1 \to C_2$ the map $\mathrm{Hom}_{D(\mathsf{C})}(C_2, K) \to \mathrm{Hom}_{D(\mathsf{C})}(C_1, K)$ is bijective. ♠

Obviously any $\mathscr{T}$-flasque chain complex is also $\mathscr{T}$-local. Some related results concerning $\mathscr{T}$-localness are summarised in the following lemma, whose proof is an exercise in fibrant and cofibrant replacements.

* Note that this differs slightly from the terminology of [CD09].

Lemma 1.4.10 ([CD09, Lemma 4.1 and Proposition 4.2]). *A chain complex K is $\mathscr{T}$-flasque if and only if it is $\mathcal{G}$-fibrant and for any $\mathscr{T}$-local equivalence $f\colon C_1 \to C_2$ the map $\mathrm{Hom}_{D(\mathsf{C})}(C_2, K) \to \mathrm{Hom}_{D(\mathsf{C})}(C_1, K)$ is bijective.*

A chain complex K is $\mathscr{T}$-local if and only if for any $T \in \mathscr{T}$ and any $n \in \mathbf{Z}$ the group $\mathrm{Hom}_{D(\mathsf{C})}(T, \Sigma^n K)$ is trivial.

The central localisation result of Cisinski and Déglise, which is based on localisation of triangulated categories combined with the approach of [Hir03], is the following:

Proposition 1.4.11 ([CD09, Proposition 4.3]). *Given a Grothendieck category C with a set of objects $\mathcal{G}$, let $\mathsf{Ch}(\mathsf{C})$ have the $\mathcal{G}$-model structure. For any set $\mathscr{T}$ of objects in $\mathsf{Ch}(\mathsf{C})$ the left Bousfield localisation with respect to the chain maps $\{0 \to \Sigma^n T \mid T \in \mathscr{T}, n \in \mathbf{Z}\}$ exist.*

In the localised model structure the weak equivalence are $\mathscr{T}$-local equivalences, while the fibrant objects are $\mathscr{T}$-local objects that are $\mathcal{G}$-fibrant, i.e. the $\mathscr{T}$-flasque objects.

The homotopy category of the left Bousfield localised model category is triangulated, and the induced functor between the homotopy categories is also triangulated.

Later we will be interested in the situation where the Grothendieck category is $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$ for some Ab-enriched category A having a small skeleton $\mathscr{A}$. In this setting the set $\mathcal{G}$ will be

$$\mathcal{G} = \{\mathrm{G} = \mathrm{Hom}_{\mathsf{A}}(G, \_)\colon \mathsf{A} \to \mathsf{Ab} \mid G \text{ an object of } \mathscr{A}\}. \tag{$\bigstar$}$$

If each set $\mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$ is free, then the fibrancy condition of Definition 1.4.9 in superfluous. To see this, note that all objects are $\mathcal{G}$-fibrant by [CD09, Proposition 2.5], Lemma 1.3.8 and Proposition 1.3.7. Consequentially all $\mathscr{T}$-local objects are also $\mathscr{T}$-flasque.

In the particular case where A has one object A with $\mathbf{Z}$ as its endomorphism ring, the above model structure reduces to the projective model structure of Example 1.4.2. For a more detailed comparison with the model structures constructed in [CH02] see [CD09, Remark 2.12].

There are other ways to give a model structure to $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$. Of particular interest is the projective model structure, which is an extension of the model structure of $\mathsf{Ch}(\mathsf{Ab})$ to the functor category. The general definition of a projective model structure on a functor category is as follows:

Definition 1.4.12. Let D be a model category and C any essentially small category*. In the *projective model structure* on $[\mathsf{C}, \mathsf{D}]$ a natural transformation $\eta\colon \mathrm{F} \to \mathrm{F}'$ is a weak equivalence (resp. fibration) if $\eta_C\colon \mathrm{F}(C) \to \mathrm{F}'(C)$ is a weak equivalence (resp. fibration) in the model structure on D for all objects C of C. ♠

In order for the above definition to have any value, the above mentioned projective model structure has to give $[\mathsf{C}, \mathsf{D}]$ a model category structure. This is indeed the case, and for particular nice model categories D the functor category inherit those nice properties.

* A category is essentially small if its skeleton is a set.

Theorem 1.4.13 ([Bar10, Theorem 2.16 and Theorem 2.18])**.** *Let* D *be a proper combinatorial model category and* C *any essentially small category. The projective model structure on* $[\mathsf{C}, \mathsf{D}]$ *is a proper combinatorial model category structure. In this model structure an object* F *is fibrant if and only if* $\mathrm{F}(A)$ *is fibrant for all* $A \in \mathsf{A}$.

The above definition and theorem also works in the Ab-enriched setting by [Lur09, Proposition A.3.3.2 and its proof]. Thus it makes sense to speak about the projective model structure on $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ for any Ab-enriched category A with a small skeleton.

One reason for working with the projective model structure on $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ is that it coincides with the $\mathcal{G}$-model structure:

Lemma 1.4.14. *The projective model structure of Barwick and the* $\mathcal{G}$*-model structure of Cisinski and Déglise coincide on* $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ *if* $\mathcal{G}$ *is defined as in* ($\bigstar$) *and the hom sets of* A *are free abelian groups.*

Proof. In both model structures on $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ the weak equivalences are the quasi-isomorphisms, so to show that the model structures coincide it is enough to show that they have the same fibrations.

To see this, note that by [CD09, Corollary 5.5] a morphism $\rho\colon \mathrm{F} \to \mathrm{F}'$ is a fibration in the $\mathcal{G}$-model structure if and only if it is levelwise $\mathcal{G}$-surjective with $\mathcal{G}$-local kernel. The surjectivity condition means that for each object G of A and $n \in \mathbf{Z}$ the induced map $\mathrm{Hom}_{[\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}}\big(\mathrm{Hom}_{\mathsf{A}}(G, \_), \mathrm{F}_n\big) \to \mathrm{Hom}_{[\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}}\big(\mathrm{Hom}_{\mathsf{A}}(G, \_), \mathrm{F}'_n\big)$ is surjective, i.e. the map $(\rho_G)_n\colon \mathrm{F}_n(G) \to \mathrm{F}'_n(G)$ is surjective. An object F of $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ is $\mathcal{G}$-local if the morphism $\mathrm{Hom}_{K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\Sigma^n \mathrm{G}^0, \mathrm{F}) \to \mathrm{Hom}_{D([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\Sigma^n \mathrm{G}^0, \mathrm{F})$ (where G^0 is the chain complex that is $\mathrm{Hom}_{\mathsf{A}}(G, \_)$ in degree 0 and 0 otherwise) is an isomorphism for all $n \in \mathbf{Z}$ and $G \in \mathsf{A}$.

Thus, if $\rho\colon \mathrm{F} \to \mathrm{F}'$ is a fibration in the $\mathcal{G}$-model structure, then ρ_B is a fibration in the projective model structure on $\mathsf{Ch}(\mathsf{Ab})$ for all objects B of A so ρ is a projective fibration. Conversely, if $\rho\colon \mathrm{F} \to \mathrm{F}'$ is a projective fibration, it only remains to show that its kernel is $\mathcal{G}$-local. However, since Lemma 1.3.8 and Proposition 1.3.7 implies that all objects are $\mathcal{G}$-fibrant, this follows by [CD09, Theorem 2.5] which identifies the $\mathcal{G}$-local objects and the $\mathcal{G}$-fibrant objects. $\square$

Note that if $\eta\colon \mathrm{K} \to \mathrm{G}$ is a cofibration, then $\eta_A\colon \mathrm{K}(A) \to \mathrm{G}(A)$ is a cofibration in the projective model structure on $\mathsf{Ch}(\mathsf{Ab})$ for all objects A in A. To see this observe that if $f\colon C_1 \to C_2$ is an acyclic fibration in the projective model structure on $\mathsf{Ch}(\mathsf{Ab})$, C_1 and C_2 are the constant functors to C_1 and C_2 respectively, and $\zeta^f_A = f$ for all objects A in A, then the morphism $\zeta^f\colon \mathrm{C}_1 \to \mathrm{C}_2$ in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ is an acyclic fibration. It follows that η_A has the left lifting property with respect to all acyclic fibrations of $\mathsf{Ch}(\mathsf{Ab})$, so η_A must be a cofibration.

If the model category C is a closed symmetric monoidal category (e.g. $\mathsf{Ch}(\mathsf{Ab})$ and $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$), then one would like the tensor product to play nicely with the model structure. The precise statement is that C ought to be a *monoidal model category*. This definition focuses on the cofibrations. For instance, if $i\colon A \rightarrowtail B$ and $j\colon C \rightarrowtail D$ are cofibrations, consider the pushout

$$\begin{tikzcd}
A \otimes C \arrow[r, "i\otimes \mathrm{id}_C"] \arrow[d, "\mathrm{id}_A\otimes j"'] \arrow[dr, phantom, "\ulcorner."] & B \otimes C \arrow[d] \\
A \otimes D \arrow[r] & (A \otimes D) \coprod_{A\otimes C} (B \otimes C)
\end{tikzcd}$$

and note that there is a evident chain map

$$(i \otimes \mathrm{id}_D) \amalg (\mathrm{id}_B \otimes j)\colon (A \otimes D) \coprod_{A\otimes C} (B \otimes C) \to B \otimes D.$$

A natural question to ask is whether this is a cofibration or not and when it is a weak equivalence. Similarly, it would be nice if the unit for the tensor product is cofibrant.

Definition 1.4.15. Suppose $(\mathsf{C}, \otimes, S, \mathrm{Hom})$ is a closed symmetric monoidal category and a model category. The model structure is *monoidal* if the following two axioms hold:

The pushout product axiom Given cofibrations $i\colon A \rightarrowtail B$ and $j\colon C \rightarrowtail D$, the induced morphism

$$(i \otimes \mathrm{id}_D) \coprod (\mathrm{id}_B \otimes j)\colon (A \otimes D) \coprod_{A\otimes C} (B \otimes C) \longrightarrow B \otimes D$$

is a cofibration. In addition, it is an acyclic cofibration if either i or j is a weak equivalence.

Cofibrant unit axiom The object S is cofibrant*. ♠

Note that $\mathsf{Ch}(\mathsf{Ab})$ has a closed monoidal structure, and by [Hov99, Theorem 4.2.13] the projective model structure on $\mathsf{Ch}(\mathsf{Ab})$ is monoidal. One reason for wanting a monoidal model structure is that it ensures that the homotopy category is also monoidal:

Theorem 1.4.16 ([Hov99, Theorem 4.3.2])**.** *Suppose* $(\mathsf{C}, \otimes, S, \mathrm{Hom})$ *is a closed symmetric category which is also a monoidal model category. The triple* $(\mathsf{Ho}(\mathsf{C}), \otimes^L, S, R\,\mathrm{Hom})$ *then gives the homotopy category the structure of a symmetric monoidal category (where* $\otimes^L$ *is the total left derived functor of* $\otimes$ *and* $R\,\mathrm{Hom}$ *is the total right derived functor of* Hom*).*

If A is a symmetric monoidal category, so is $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$. Moreover, the existence of a monoidal model structure is a question of cofibrations and acyclic cofibrations. In the projective model category structure the generating cofibrations and acyclic cofibrations are known (see for instance [Lur09, Remark A.3.3.5 and the proof of Proposition A.2.8.2]), so it is not hard to show that it gives a monoidal model structure.

* A more general version of this axiom can be found in [Hov99, Definition 4.2.6]

Proposition 1.4.17. *The projective model structure on* $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ *is monoidal.*

Proof. The generating cofibrations of the projective model structure are on the form

$$\iota_A^n\colon \mathbf{S}^{n-1}\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big) \to \mathbf{D}^n\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big)$$

for some object A in A and integer n. Fix objects A and B of A and integers n, m. Since $\underline{\otimes}$ is an extension of the tensor product on A, there is an isomorphism

$$\mathrm{S} = \mathbf{S}^{n-1}\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big)\underline{\otimes}\mathbf{S}^{m-1}\big(\operatorname{Hom}_{\mathsf{A}}(B, \_)\big) \simeq \mathbf{S}^{n+m-2}\big(\operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_)\big),$$

and the isomorphism

$$\mathrm{D}_+ = \mathbf{D}^{n}\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big)\underline{\otimes}\mathbf{S}^{m-1}\big(\operatorname{Hom}_{\mathsf{A}}(B, \_)\big) \simeq \mathbf{D}^{n+m-1}\big(\operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_)\big)$$

follows by a similar computation. For the "opposite" case, the same technique is applicable, but now the integer n must be taken into consideration. Thus

$$\mathrm{D}_- = \mathbf{S}^{n-1}\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big)\underline{\otimes}\mathbf{D}^{m}\big(\operatorname{Hom}_{\mathsf{A}}(B, \_)\big) \simeq (-1)^{n-1}\mathbf{D}^{n+m-1}\big(\operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_)\big)$$

where $(-1)^p\mathbf{D}^q(\mathrm{F})$ is degree-wise isomorphic to $\mathbf{D}^q(\mathrm{F})$ while the only non-zero boundary morphism is $(-1)^p\mathrm{id}_{\mathrm{F}}$. It follows that $\mathrm{D}_+ \coprod_{\mathrm{S}} \mathrm{D}_- \simeq \mathbf{D}^{n+m-1}\big(\operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_)\big)$ Lastly, if $\mathrm{D} = \mathbf{D}^n\big(\operatorname{Hom}_{\mathsf{A}}(A, \_)\big)\underline{\otimes}\mathbf{D}^m\big(\operatorname{Hom}_{\mathsf{A}}(B, \_)\big)$ then

$$\mathrm{D}_k \simeq \begin{cases} \operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_) & k = n+m, k = n+m-2, \\ \operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_) \oplus \operatorname{Hom}_{\mathsf{A}}(A\otimes_\sigma B, \_) & k = n+m-1, \\ 0 & \text{otherwise}, \end{cases}$$

where the non-zero boundary morphisms ∂_k are given by $(\partial_{n+m})_C(f) = \big((-1)^n f, f\big)$ and $(\partial_{n+m-1})_C(f_1, f_2) = f_1 + (-1)^{n-1} f_2$ for C an object of A. The induced morphism $I\colon (\mathrm{D}_+ \coprod_{\mathrm{S}} \mathrm{D}_-) \to \mathrm{D}$ is given by the identity in degree $n+m-2$, while $(I_{n+m-1})_C(f) = (f, (-1)^n f)$.

Let $\rho\colon \mathrm{K} \to \mathrm{L}$ be an acyclic fibration in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$, and consider a commutative diagram

$$\begin{tikzcd} \mathrm{D}_+ \coprod_{\mathrm{S}} \mathrm{D}_- \arrow[r] \arrow[d, "I"] & \mathrm{K} \arrow[d, "\rho"] \\ \mathrm{D} \arrow[r] & \mathrm{L}. \end{tikzcd}$$

If one apply the entire diagram to $A\otimes_\sigma B$, the morphism $I_{A\otimes_\sigma B}$ will be a cofibration in $\mathsf{Ch}(\mathsf{Ab})$. Thus there is a lift $\alpha_{A\otimes_\sigma B}\colon \mathrm{D}(A\otimes_\sigma B) \to \mathrm{K}(A\otimes_\sigma B)$. In each degree this lift can be extended to natural transformations $(\alpha_C)_k\colon \mathrm{D}(C)_k \to \mathrm{K}(C)_k$ by defining $(\alpha_C)_k(f) = \mathrm{K}_k(f)\big((\alpha_{A\otimes_\sigma B})_k(\mathrm{id}_{A\otimes_\sigma B})\big)$ for $k = n+m$ and $k = n+m-2$, while

$$\begin{aligned}(\alpha_C)_{n+m-1}(f_1, f_2) = {}& \mathrm{K}_{n+m-1}(f_1)_{n+m-1}\big((\alpha_{A\otimes_\sigma B})_{n+m-1}(\mathrm{id}_{A\otimes_\sigma B})\big) \\ &+ (-1)^{n-1}\mathrm{K}_{n+m-1}(f_2)_{n+m-1}\big((\alpha_{A\otimes_\sigma B})_{n+m-1}(\mathrm{id}_{A\otimes_\sigma B})\big).\end{aligned}$$

Since $\alpha_{A\otimes_\sigma B}$ is a chain map, so is α_C. Thus there is a lift in the original diagram, and since this holds for any acyclic fibration I is a cofibration in the projective model structure on $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$. Since the projective cofibrations are obtained from the generating cofibrations by pushouts, transfinite compositions and retracts it follows that the first part of the pushout product axiom is satisfied. The second part follows by a similar computation.

Since the unit for the tensor product on $\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})$ is given by $\mathrm{Hom}_{\mathsf{A}}(S,\_)^0$ where S is the unit for $\otimes_\sigma$, the same technique as above shows that the cofibrant unit axiom holds. $\square$

Note that left Bousfield localisation leave the cofibrations alone so to check that the localised model structures are monoidal, it is enough to check the part about acyclic cofibrations. If the localisation is done by Proposition 1.4.11, this reduces to a question about flatness – see Proposition 1.4.19.

Definition 1.4.18. Given a symmetric monoidal Grothendieck category $(\mathsf{C},\otimes,S)$ with a set of objects $\mathcal{G}$, then $\mathcal{G}$ is a *flat family* if

- All the objects of $\mathcal{G}$ are flat*.
- If $G_1, G_2 \in \mathcal{G}$ then $G_1 \otimes G_2 \in \mathcal{G}$.
- The unit for the tensor product, S, is in $\mathcal{G}$.

If $\mathcal{G}$ is a flat family, then a collection $\mathscr{T}$ of $\mathcal{G}$-cofibrant objects in $\mathsf{Ch}(\mathsf{C})$ is called *flat* if for each $\mathrm{G} \in \mathcal{G}$ and $\mathrm{T} \in \mathscr{T}$ the object $\mathrm{T} \otimes_T G^0$ is in $\mathscr{T}$ (here G^0 is the object that is G in degree 0 and 0 otherwise). ♠

In the case where $\mathsf{C} = [\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}$, $\mathcal{G}$ is defined as in equation (★) and the hom sets of A are free abelian groups, the set $\mathcal{G}$ is a flat family. To see this, note that by Lemma 1.3.8 all objects of $\mathcal{G}$ are projective, and hence flat. Moreover, the unit for the internal tensor product on $[\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}}$ is $\mathrm{Hom}_{\mathsf{A}}(S,\_)$ so it is in $\mathcal{G}$ (where the object $S \in \mathsf{A}$ is the unit for $\otimes_\sigma$). Lastly, since the internal tensor product extends $\otimes_\sigma$, it is clear that $\mathcal{G}$ is closed under tensor products.

Proposition 1.4.19 ([CD09, Proposition 3.7 and Corollary 4.11])**.** *If $\mathsf{Ch}(\mathsf{C})$ has the $\mathcal{G}$-model structure for a flat family $\mathcal{G}$ and $\mathscr{T}$ is a flat set of objects in $\mathsf{Ch}(\mathsf{C})$ then then left Bousfield localisation obtained by Proposition 1.4.11 gives a monoidal model category and the induced functor between homotopy categories is monoidal*†.

Since a model category is a model for its homotopy category, it is of interest to know if there are other models for the same homotopy category. The naïve way is to say that if two model categories D and E have isomorphic homotopy categories, then they are the same model. However, this does not consider the fibration and cofibration data of the model structure.

* I.e. if $G \in \mathcal{G}$ then $G \otimes \_$ is exact. † Even more is true: The localised model category also satisfies the monoid axiom of [SS00].

Definition 1.4.20. Suppose D and E are model categories, and that

$$\mathrm{F}\colon \mathsf{D} \rightleftarrows \mathsf{E}\colon \mathrm{U}$$

is an adjunction with U right adjoint to F. If F preserves cofibrations and acyclic cofibrations, then F is a *left Quillen functor*. Dually, if U preserves fibrations and acyclic fibrations, then U is a *right Quillen functor*. In the case where F is a left Quillen functor and U is a right Quillen functor, the adjunction is a *Quillen adjunction*. ♠

As with the definition of a model category, the definition of a Quillen adjunction is "over-determined". To see this, let $\mathrm{F}\colon \mathsf{D} \rightleftarrows \mathsf{E}\colon \mathrm{U}$ be an adjunction of model categories. If $i\colon D_1 \to D_2$ and $p\colon E_1 \to E_2$ are morphisms in D and E respectively, then $\mathrm{F}(i)$ has the left lifting property with respect to p if and only if $\mathrm{U}(p)$ has the right lifting property with respect to i. Thus F is a left Quillen functor if and only if U is a right Quillen functor.

For C a model category, let C_c (resp. C_f) be the full subcategory of cofibrant (resp. fibrant) objects. Both C_c and C_f inherits a notion of weak equivalences from C. Moreover, by MC2, MC4 and MC5 there is a functor $\mathrm{Q}\colon \mathsf{C} \to \mathsf{C}_c$ (resp. $\mathrm{R}\colon \mathsf{C} \to \mathsf{C}_f$) called cofibrant replacement (resp. fibrant replacement). Since the cofibrant (resp. fibrant) replacement takes an object C to a cofibrant (resp. fibrant) object that is weakly equivalent to C, it follows that the induced functors on the homotopy categories are equivalences.

Given a functor $\mathrm{F}\colon \mathsf{C} \to \mathsf{C}'$ with C a model category, it will induce a functor $\mathsf{Ho}(\mathsf{C}) \to \mathsf{C}'$ if it takes weak equivalences to isomorphisms. It follows that if C' has a collection of morphisms called weak equivalences, then a sufficient criterion for an induced functor $\mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{C}')$ is that F takes weak equivalences to weak equivalences. However, in the case where C' is a model category, this is *not* a necessary requirement.

Consider the case where where C' is a model category and F is a left Quillen functor. In this situation one has control over the acyclic cofibrations. Combining this with the equivalence $\mathsf{Ho}(\mathsf{C}) \to \mathsf{Ho}(\mathsf{C}_c)$ forms the basis of another another aspect of Quillen adjunctions, namely that Quillen adjunctions descends to an adjunction on the homotopy categories. A proof of this fact using the using the notion of path and cylinder objects in a model category can be found in [Hov99]:

Proposition 1.4.21 ([Hov99, Lemma 1.3.10])**.** *Suppose* D *and* E *are model categories and*

$$\mathrm{F}\colon \mathsf{D} \rightleftarrows \mathsf{E}\colon \mathrm{U} \qquad (*)$$

is a Quillen adjunction. Denote by $\mathrm{LF}\colon \mathsf{Ho}\big(\mathsf{D}\big) \to \mathsf{Ho}\big(\mathsf{E}\big)$ *the composition* $\mathsf{Ho}(\mathsf{D}) \to \mathsf{Ho}(\mathsf{D}_c) \to \mathsf{Ho}(\mathsf{E})$ *and by* $\mathrm{RU}\colon \mathsf{Ho}\big(\mathsf{E}\big) \to \mathsf{Ho}\big(\mathsf{E}\big)$ *the composition* $\mathsf{Ho}(\mathsf{E}) \to \mathsf{Ho}(\mathsf{E}_f) \to \mathsf{Ho}(\mathsf{E})$*. The Quillen adjunction* $(*)$ *induces an adjunction*

$$\mathrm{LF}\colon \mathsf{Ho}(\mathsf{D}) \rightleftarrows \mathsf{Ho}(\mathsf{E})\colon \mathrm{RU}.$$

The functor LF (resp. RU) constructed in Proposition 1.4.21 is the *total left derived functor* of F (resp. the *total right derived functor* of U).

Definition 1.4.22. Suppose D and E are model categories and

$$\mathrm{F}\colon \mathsf{D} \rightleftarrows \mathsf{E}\colon \mathrm{U} \tag{$*$}$$

is a Quillen adjunction. If the *derived adjunction*

$$\mathrm{LF}\colon \mathsf{Ho}(\mathsf{D}) \rightleftarrows \mathsf{Ho}(\mathsf{E})\colon \mathrm{RU}$$

is an equivalence of categories, then the adjunction $(*)$ is called a *Quillen equivalence.*

A *zig-zag of Quillen equivalences* from D to E is a finite set of model categories $\mathsf{C}_0, \dots, \mathsf{C}_n$ such that $\mathsf{C}_0 = \mathsf{D}$, $\mathsf{C}_n = \mathsf{E}$ and for $i = 0, \dots, n-1$ there is a Quillen equivalence between C_i and C_{i+1}. In such a case the two categories D and E are *Quillen equivalent.* ♠

As a final observation, if there is an adjunction between D and E, then there is an induced adjunction between $[\mathsf{C}, \mathrm{D}]$ and $[\mathsf{C}, \mathrm{E}]$. Moreover, this extends to the setting of Quillen adjunctions.

Lemma 1.4.23. *Let C be a category, and*

$$\mathrm{F}\colon \mathsf{D} \rightleftarrows \mathsf{E}\colon \mathrm{U} \tag{$*$}$$

an adjunction. The adjunction in $(*)$ extends to an adjunction

$$\mathrm{F}_*\colon [\mathsf{C}, \mathsf{D}] \rightleftarrows [\mathsf{C}, \mathsf{E}]\colon \mathrm{U}_*. \tag{$**$}$$

Moreover, if D and E are model categories and U is a right Quillen functor, then U_* is a right Quillen functor when both $[\mathsf{C}, \mathsf{D}]$ and $[\mathsf{C}, \mathsf{E}]$ have the projective model structure.

Proof. By using the unit $\eta\colon \mathrm{id}_{\mathsf{E}} \to \mathrm{U} \circ \mathrm{F}$ and counit $\epsilon\colon \mathrm{F} \circ \mathrm{U} \to \mathrm{id}_{\mathsf{D}}$ it is straightforward to see that $\eta'\colon \mathrm{id}_{[\mathsf{C},\mathsf{E}]} \to \mathrm{U}_*\mathrm{F}_*$ with $\eta'_{\mathrm{V}}(C) = \eta_{\mathrm{V}(C)}$ and $\epsilon'\colon \mathrm{F}_*\mathrm{U}_* \to \mathrm{id}_{[\mathsf{C},\mathsf{D}]}$ with $\epsilon'_{\mathrm{W}}(C) = \epsilon_{\mathrm{W}(C)}$ gives a unit/counit pair for the adjunction in $(**)$.

Let W_1, W_2 be objects of $[\mathsf{C}, \mathsf{D}]$ and $\zeta\colon \mathrm{W}_1 \to \mathrm{W}_2$ a natural transformation. If ζ is a fibration (resp. acyclic fibration) then ζ_C is a fibration (resp. acyclic fibration) for all objects C of C. Since U preserves fibrations (resp. acyclic fibrations) $\mathrm{U}(\zeta_C) = \mathrm{U}_*(\zeta)_C$ is also a fibration (resp. acyclic fibration), so $\mathrm{U}_*(\zeta)$ is a fibration (resp. acyclic fibration). □

2 Application to operator algebras

The aim of this chapter is to use the machinery of the previous chapter to create a suitable model category for working with KK-theory. Moreover, the resulting model category will be compared to the work in [Øst10].

2.1 Adapting the machinery

The aim of this section is to adapt the earlier machinery to operator algebras. First note that earlier, the "source" category was required to be Ab-enriched. This is a slight problem since the category of C*-algebras is not Ab-enriched. However, the next definition solves the problem:

Definition 2.1.1. If C is a category, then the category C_{Ab} has the same objects as C, while for objects C_1, C_2 of C the set $\mathrm{Hom}_{\mathsf{C}_{\mathsf{Ab}}}(C_1, C_2) = \mathbf{F}\,\mathrm{Hom}_{\mathsf{C}}(C_1, C_2)$ of morphisms from C_1 to C_2 in C is the free abelian group generated by the morphisms from C_1 to C_2 in C. Define composition in C_{Ab} by extending the composition in C by linearity.

If C is pointed (i.e. has a zero object), the category $\mathsf{C}_{\mathsf{Ab}^0}$ has the same objects as C_{Ab}, but $\mathrm{Hom}_{\mathsf{C}_{\mathsf{Ab}^0}}(C_1, C_2) = \mathrm{Hom}_{\mathsf{C}_{\mathsf{Ab}}}(C_1, C_2)/\{z = 0\}$ where $z\colon C_1 \to C_2$ is the zero morphism. ♠

Since all categories C considered in this thesis are locally small, it is straightforward to see that C_{Ab} is an Ab-enriched category. Moreover, the obvious inclusion of C into C_{Ab} is a faithful functor. If C is pointed, then this also holds for $\mathsf{C}_{\mathsf{Ab}^0}$. Note that the category C_{Ab} inherits a universal property from free abelian groups:

Lemma 2.1.2. *If* A *is* Ab*-enriched and* $\mathrm{F}\colon \mathsf{C} \to \mathsf{A}$ *is a functor, then there exists an unique* Ab*-functor* $\mathrm{F}_{\mathsf{Ab}}\colon \mathsf{C}_{\mathsf{Ab}} \to \mathsf{A}$ *such that*

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "\mathrm{F}"] \arrow[dr] & & \mathsf{A} \\
& \mathsf{C}_{\mathsf{Ab}} \arrow[ur, dashed, "\mathrm{F}_{\mathsf{Ab}}"'] &
\end{tikzcd}$$

commutes. If C *is pointed and* F *takes basepoints to zero morphisms, then the analogous statement for* $\mathsf{C}_{\mathsf{Ab}^0}$ *holds.*

Proof. By the universal property of free abelian groups, the functor F induces a functor $\mathrm{F}_{\mathsf{Ab}}\colon \mathsf{C}_{\mathsf{Ab}} \to \mathsf{A}$ which is the identity on objects, while a morphism $\sum_{i=1}^{M} n_i f_i\colon C_1 \to C_2$ in C_{Ab} is sent to $\sum_{i=1}^{M} n_i \mathrm{F}(f_i) \in \mathrm{Hom}_{\mathsf{A}}\big(\mathrm{F}(C_1), \mathrm{F}(C_2)\big)$. Uniqueness of the functor follows from the uniqueness part of the universal property of free abelian groups. □

The "extension by linearity" approach of Lemma 2.1.2 is both functorial and invertible, as Lemma 2.1.3 shows.

Lemma 2.1.3. *Let* C *be a category and* A *an* Ab*-enriched category. Extension by linearity gives an isomorphism of categories* $\__{\mathsf{Ab}}\colon [\mathsf{C},\mathsf{A}] \to [\mathsf{C}_{\mathsf{Ab}},\mathsf{A}]_{\mathsf{Ab}}$.

Proof. For the functorial part it remains to show that the construction of Lemma 2.1.2 takes natural transformations to natural transformations. To see this, note that if $\eta\colon \mathrm{F} \to \mathrm{G}$ is a natural transformation between functors $\mathsf{C} \to \mathsf{A}$, then for any $f\colon C_1 \to C_2$ in C there is a commutative square

$$\begin{tikzcd}
\mathrm{F}(C_1) \arrow[r, "\eta_{C_1}"] \arrow[d, "\mathrm{F}(f)"] & \mathrm{G}(C_1) \arrow[d, "\mathrm{G}(f)"] \\
\mathrm{F}(C_2) \arrow[r, "\eta_{C_2}"] & \mathrm{G}(C_2)
\end{tikzcd}$$

in A. Thus η is well-behaved on the generators of $\mathrm{Hom}_{\mathsf{C}_{\mathsf{Ab}}}(C_1, C_2)$, so η extends to a natural transformation $\eta_{\mathsf{Ab}}\colon \mathrm{F}_{\mathsf{Ab}} \to \mathrm{G}_{\mathsf{Ab}}$.

For the isomorphism claim, let $\mathrm{Z}\colon \mathsf{C} \to \mathsf{C}_{\mathsf{Ab}}$ be the inclusion functor and consider $\mathrm{Z}^*\colon [\mathsf{C}_{\mathsf{Ab}},\mathsf{A}]_{\mathsf{Ab}} \to [\mathsf{C},\mathsf{A}]$, $\mathrm{F} \mapsto \mathrm{F} \circ \mathrm{Z}$. Clearly $\mathrm{F}_{\mathsf{Ab}} \circ \mathrm{Z} = \mathrm{F}$ and $(\mathrm{H} \circ \mathrm{Z})_{\mathsf{Ab}} = \mathrm{H}$ for $\mathrm{F}\colon \mathsf{C} \to \mathsf{A}$ and $\mathrm{H} \in [\mathsf{C}_{\mathsf{Ab}},\mathsf{A}]_{\mathsf{Ab}}$, whence there is an equivalence of categories. $\square$

Consider the category $\mathsf{C}^*\text{-}\mathsf{alg}$ with separable C*-algebras as objects and morphisms the $*$-preserving algebra homomorphisms. This is a pointed category, so $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ is Ab-enriched*. Note that the algebraic tensor product of two C*-algebras does not need to be a C*-algebra. However, it is always possible to find a norm such that the completion of the algebraic tensor product in this norm is a C*-algebra. In the special case where the C*-algebra A is *nuclear*, there is a unique C*-algebra norm on $A \otimes B$ for any C*-algebra B. Examples of nuclear C*-algebras are the the commutative C*-algebras, the matrix algebras over $\mathbf{C}$ and the C*-algebra of compact operators on ℓ^2.

For other C*-algebras one has to choose a norm. Among the norms to choose from are the minimal norm and the maximal norm (in fact, all C*-algebra norms must "lie between" the minimal norm and the maximal norm – see [Weg93, Appendix T]).

Definition 2.1.4. Given two C*-algebra A_1 and A_2, their *minimal tensor product* or *spatial tensor product* is the C*-algebra $A_1 \otimes_\sigma A_2$, which is the completion of the algebra $A_1 \otimes A_2$ in the norm

$$\|x\| = \sup \|(\pi_1 \otimes \pi_2)(x)\|$$

where π_1, π_2 runs over all representations of A_1 and A_2 respectively.

The extension of $\otimes_\sigma$ to $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ by linearity will also be denoted by $\otimes_\sigma$. Since $\otimes_\sigma$ makes $\mathsf{C}^*\text{-}\mathsf{alg}$ into a symmetric monoidal category, it follows that $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ is also a symmetric monoidal category. ♠

* Since Ab has a biproduct, $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ is in fact an additive category.

The "obvious" functor from C*-algebras to $[\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}$ is the functor YZ, which is the inclusion of $\mathsf{C^*\text{-}alg}$ into $\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$ followed by the contravariant Yoneda embedding $A \mapsto \mathrm{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}}(A, \_)$.

Now form the category $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ and notice that this category has both an internal hom object and an internal tensor product (the details can be found in Section 1.2). Note that for any Ab-enriched category A there is an Ab-functor $\_^0 \colon \mathsf{A} \to \mathsf{Ch}(\mathsf{A})$ which sends an object A to the object A^0 which is A in degree 0 and elsewhere 0. Combining this with the discussion above gives a functor from C*-algebras to $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$:

Definition 2.1.5. The functor $[\_]^{\mathrm{Z}} \colon \mathsf{C^*\text{-}alg}^{\mathrm{op}} \to \mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is obtained by composing the functors YZ and $\_^0$. Thus $[A]^{\mathrm{Z}}$ is $\mathrm{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}}(A, \_)$ in degree 0 and elsewhere 0. ♠

From Section 1.4 $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is a monoidal model category. The trouble with this model structure is that its homotopy category has little with the traditional view of homotopy in C*-algebras. For instance, homotopic maps do not necessarily become the same map in the homotopy category. To remedy this situation Bousfield localisation will be used, and this is the aim of Section 2.2.

The motivation for the localisations are KK-theory, which is a C*-algebraic unification of K-theory and K-homology. This unified theory is given by the functor $\mathrm{KK} \colon \mathsf{C^*\text{-}alg}^{\mathrm{op}} \times \mathsf{C^*\text{-}alg} \to \mathsf{Ab}$. There are several pictures of the functor KK, and the one on display in Definition 2.1.6 is by Cuntz[Cun87].

Definition 2.1.6. Denote by $A * B$ the free product* of A and B, let qA be the kernel of the $*$-homomorphism $\mathrm{id}_A * \mathrm{id}_A \colon A * A \to A$. For separable C*-algebras A and B, the abelian group $\mathrm{KK}(A, B)$ is given as $\mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(qA, \mathbb{K} \otimes B)/\sim$, where $f \sim g$ if f and g are homotopic. The group operation of $\mathrm{KK}(A, B)$ is given by

$$\langle f \rangle + \langle g \rangle = \left\langle M_B \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \right\rangle$$

where $M_B \colon \mathrm{Mat}_2(B \otimes_\sigma \mathbb{K}) \simeq B \otimes_\sigma \mathrm{Mat}_2(\mathbb{K}) \to B \otimes_\sigma \mathbb{K}$, $b \otimes x \mapsto b \otimes M(x)$ for some fixed isomorphism $M \colon \mathrm{Mat}_2(\mathbb{K}) \to \mathbb{K}$. ♠

One property of KK is that it gives rise to an additive category KK. In this category the objects are C*-algebras while $\mathrm{Hom}_{\mathsf{KK}}(A_1, A_2) = \mathrm{KK}(A_1, A_2)$. Moreover, the functoriality of KK gives a functor $\mathrm{KK} \colon \mathsf{C^*\text{-}alg} \to \mathsf{KK}$. This functor encapsulates KK-theory, in the sense that it enjoys the universal property that characterises KK-theory.

Proposition 2.1.7 ([Bla98, Corollary 22.3.1])**.** *Any homotopy invariant (see Definition 2.2.3) and matrix stable (see Definition 2.2.10) functor* $\mathrm{F} \colon \mathsf{C^*\text{-}alg} \to \mathsf{Ab}$ *with the coproduct property (see Definition 2.2.7) factors through* KK*, i.e. there is a functor* $\mathrm{F}' \colon \mathsf{KK} \to \mathsf{Ab}$ *such that* $\mathrm{F} = \mathrm{F}' \circ \mathrm{KK}$*.*

* I.e. the completion of the algebraic free product in the norm $\|x\| = \sup \|\pi(x)\|$ where π runs over all representations of $A * B$. Note that this gives the coproduct in the category of C*-algebras.

Note that KK induces a functor $\mathrm{KK}^{\mathrm{op}} \colon \mathsf{C^*\text{-}alg}^{\mathrm{op}} \to \mathsf{KK}^{\mathrm{op}}$, and this has a similar universal property as the one of Proposition 2.1.7. Namely if $\mathrm{F} \colon \mathsf{C^*\text{-}alg}^{\mathrm{op}} \to \mathsf{Ab}$ is homotopy invariant, matrix stable and has the coproduct property* then F factors through $\mathsf{KK}^{\mathrm{op}}$. Thus KK-theory also has a universal property with respect to contravariant functors.

Note that for KK-theory the minimal tensor product is the appropriate one. One reason for this is that the minimal tensor product preserves split exact sequences, while KK-theory in some sense is the study of extensions in $\mathsf{C^*\text{-}alg}$.

2.2 Three localisations

The aim of this section is to endow $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ with a model structure related to KK-theory. To begin with, give $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ the projective model structure (also known as the $\mathcal{G}$-model structure). The next step is to add weak equivalences so that the category satisfies homotopy invariance, split exactness, and matrix stability. This will be done by utilising Proposition 1.4.11.

Note that localisation requires a bit of smallness. Thus fix a skeleton $\mathscr{A}$ of $\mathsf{C^*\text{-}alg}$, and note that $\mathscr{A}$ is a *set*. Moreover, the set $\mathcal{G}$, which was used for the monoidal part of the localised model structure, was given by

$$\mathcal{G} = \{\mathrm{G} = \mathrm{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}}(A, \_) \colon \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0} \to \mathsf{Ab} \mid A \text{ is an element of } \mathscr{A}\}.$$

If one is careless it is possible that too many morphisms becomes weak equivalences after a Bousfield localisation. A way to ensure that this is not the case, is to look at local objects. Recall that if $\mathrm{F} \in \mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is $\mathscr{T}$-local, then for any $\mathscr{T}$-local equivalence $\eta \colon \mathrm{C}_1 \to \mathrm{C}_2$ the induced map

$$\mathrm{Hom}_{D([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{C}_2, \mathrm{F}) \to \mathrm{Hom}_{D([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{C}_1, \mathrm{F})$$

is a bijection. This implies† that if $\mathrm{KK}(C, \_)^0_{\mathsf{Ab}^0}$ is $\mathscr{T}$-local for a C*-algebra C and $[A]^{\mathrm{Z}} \to [B]^{\mathrm{Z}}$ is a $\mathscr{T}$-local equivalence, then there is a bijection $\mathrm{KK}(C, B) \to \mathrm{KK}(C, A)$. In order to prove that $\mathrm{KK}(C, \_)^0_{\mathsf{Ab}^0}$ is $\mathscr{T}$-local in the various localisations, Lemma 2.2.2 will be used. The proof of this lemma uses the following special case of the Yoneda lemma:

Lemma 2.2.1. *Let* A *be an* Ab*-enriched category and* $\mathrm{F} \colon \mathsf{A} \to \mathsf{Ab}$ *an* Ab*-functor. If* $f \colon A_0 \to A_1$ *is a morphism of* A *then the diagram*

$$\begin{tikzcd} \mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{A}}(A_0, \_), \mathrm{F}\big) \arrow[r, "\mathrm{YF}_{A_0}"] \arrow[d, "f^{**}"] & \mathrm{F}(A_0) \arrow[d, "\mathrm{F}(f)"] \\ \mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{A}}(A_1, \_), \mathrm{F}\big) \arrow[r, "\mathrm{YF}_{A_1}"] & \mathrm{F}(A_1) \end{tikzcd}$$

* Note that a functor $\mathsf{C^*\text{-}alg}^{\mathrm{op}} \to \mathsf{Ab}$ is equivalent to a functor $\mathsf{C^*\text{-}alg} \to \mathsf{Ab}^{\mathrm{op}}$, and for an additive category the coproduct coincide with the product. † See the proof of Proposition 2.2.13.

commutes, where the leftmost vertical map, f^{**}, is $\eta \mapsto \eta \circ f^*$ while

$$\mathrm{YF}_A\colon \mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{A}}(A,\_), \mathrm{F}\big) \to \mathrm{F}(A), \qquad \eta \mapsto \eta_A(\mathrm{id}_A)$$

is an isomorphism by the Yoneda lemma. In particular, $\mathrm{F}(f)$ is injective if and only if f^{**} is injective.

Proof. The diagram is commutative, for if $\eta \in \mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{A}}(A_0,\_), \mathrm{F}\big)$ then

$$\begin{aligned}\big(\mathrm{YF}_{A_1} \circ f^{**}\big)(\eta) = \mathrm{YF}_{A_1}\big(\eta \circ f^*\big) = \big(\eta \circ f^*\big)_{A_1}(\mathrm{id}_{A_1}) = \eta_{A_1}(f) = \\ \big(\eta_{A_1} \circ f_*\big)(\mathrm{id}_{A_0}) = \big(\eta_{A_1} \circ \mathrm{Hom}_{\mathsf{A}}(A_0,\_)(f)\big)(\mathrm{id}_{A_0}) = \\ \big(\mathrm{F}(f) \circ \eta_{A_0}\big)(\mathrm{id}_{A_0}) = \big(\mathrm{F}(f) \circ \mathrm{YF}_{A_0}\big)(\eta).\end{aligned} \qquad \square$$

Note that the above lemma also holds when $\mathrm{F}\colon \mathsf{C} \to \mathsf{Set}$ is some ordinary functor. The next lemma, which is a variation of the theme "the mapping cone of an isomorphism is contractible", will be used with $\mathsf{A} = \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$ and F functors similar to $\mathrm{KK}(C,\_)$ for some C*-algebra C.

Lemma 2.2.2. *Let A be an Ab-enriched category and $\mathrm{F}\colon \mathsf{A} \to \mathsf{Ab}$ an Ab-functor. Moreover, let $f\colon A_0 \to A_1$ in A be such that $\mathrm{F}(f)$ is an isomorphism. Denote by $\mathrm{A} = (\mathrm{A}_i, \delta_i)$ the object in $\mathsf{Ch}([\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}})$ where*

$$\mathrm{A}_i = \begin{cases} \mathrm{Hom}_{\mathsf{A}}(A_1,\_) & i = 1, \\ \mathrm{Hom}_{\mathsf{A}}(A_0,\_) & i = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad \delta_i = \begin{cases} f^* & i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

1. *For any $i \in \mathbf{Z}$ the group*
$$\mathrm{Hom}_{K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{A}, \Sigma^i \mathrm{F}^0)$$
is trivial.
2. *If in addition A is monoidal and there is a natural isomorphism*
$$\mathrm{F}(\mathrm{id}\_\otimes_\sigma f)\colon \mathrm{F}(\_\otimes_\sigma A_0) \to \mathrm{F}(\_\otimes_\sigma A_1)$$
then for any Ab-functor $\mathrm{G}\colon \mathsf{A} \to \mathsf{Ab}$ and $i \in \mathbf{Z}$ the group
$$\mathrm{Hom}_{K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{G}^0 \underline{\otimes} \mathrm{A}, \Sigma^i \mathrm{F}^0)$$
is also trivial.

Proof. Statement (1): The statement obviously holds if $i \neq 0, 1$. Thus there are only two cases that need to be checked.
The case $i = 0$: Since the group $\mathrm{Hom}_{K([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{A}, \mathrm{F}^0)$ is a quotient of the group $\mathrm{Hom}_{\mathsf{Ch}([\mathsf{A},\mathsf{Ab}]_{\mathsf{Ab}})}(\mathrm{A}, \mathrm{F}^0)$, it is enough to check that the latter is trivial. Note that a morphism $\eta\colon \mathrm{A} \to \mathrm{F}^0$ is determined by a natural transformation $\eta_0\colon \mathrm{A}_0 = \mathrm{Hom}_{\mathsf{A}}(A_0,\_) \to \mathrm{F}$ such that $\eta_0 \circ f^*$ is the zero natural transformation. So $\mathrm{YF}_{A_1}(\eta_0 \circ f^*) = \mathrm{YF}_{A_1} \circ f^{**}(\eta_0) = 0$,

whence $\big(\mathrm{F}(f) \circ \mathrm{YF}_{A_0}\big)(\eta_0) = 0$ by Lemma 2.2.1. By assumption $\mathrm{F}(f)$ is injective, so $\mathrm{YF}_{A_0}(\eta_0) = 0$. Since YF_{A_0} is an isomorphism this implies that η_0 is the zero natural transformation.

The case $i = 1$: Similarly, a morphism $\eta\colon \mathrm{A} \to \Sigma^1\mathrm{F}^0$ is determined by a natural transformation $\eta_1\colon \mathrm{A}_1 = \mathrm{Hom}_{\mathsf{A}}(A_1, \_) \to \mathrm{F}$. Consider the natural transformation $\sigma_0\colon \mathrm{A}_0 = \mathrm{Hom}_{\mathsf{A}}(A_0, \_) \to \mathrm{F}$ given by $\sigma_0 = \mathrm{YF}_{A_0}^{-1} \circ \mathrm{F}(f)^{-1} \circ \mathrm{YF}_{A_1}(\eta_1)$. Then $\mathrm{F}(f) \circ \mathrm{YF}_{A_0}(\sigma) = \mathrm{YF}_{A_1}(\eta_1)$ whence $\eta_1 = f^{**}(\sigma_0) = \sigma_0 \circ f^*$. The fact that F^0 is zero except in degree 0 shows that η_1 is null-homotopic by considering the collection $\{\sigma_n\}$ where σ_n is the zero natural transformation for $n \neq 0$.

Statement (2): By the adjunction of Proposition 1.2.13 any natural transformation $\eta'\colon \mathrm{G}^0\underline{\otimes}\mathrm{A} \to \Sigma^i\mathrm{F}^0$ corresponds to a natural transformation $\eta\colon \mathrm{G}^0 \to \underline{\mathrm{Hom}}(\mathrm{A}, \Sigma^i\mathrm{F}^0)$. Since G^0 is non-zero only in degree 0, η is determined by $\eta_0\colon \mathrm{G} \to \underline{\mathrm{Hom}}(\mathrm{A}, \Sigma^i\mathrm{F}^0)_0$. Recall that for $B \in \mathsf{A}$ and $j \in \mathbf{Z}$

$$\begin{aligned}\underline{\mathrm{Hom}}\big(\mathrm{A}, \Sigma^i\mathrm{F}^0\big)(B)_j &= \mathrm{Nat}\,\big(\mathrm{O} \circ \mathrm{A}(\_), \mathrm{O} \circ \Sigma^{-j} \circ \Sigma^i \circ \mathrm{F}^0(B\otimes_\sigma\_)\big)\\ &= \mathrm{Nat}\,\big(\mathrm{O} \circ \mathrm{A}(\_), \mathrm{O} \circ \Sigma^{i-j} \circ \mathrm{F}^0(B\otimes_\sigma\_)\big)\\ &\simeq \mathrm{Nat}\,\big(\mathrm{A}_{i-j}, \mathrm{F}(B\otimes_\sigma\_)\big).\end{aligned}$$

It follows that if $j = 0$ and $i \neq 0, 1$, then this set contains only the zero natural transformation. Denote by $\underline{\mathrm{Hom}}'$ (resp. $\underline{\otimes}'$) the internal hom object (resp. internal tensor product) of $[\mathsf{A}, \mathsf{Ab}]_{\mathsf{Ab}}$. If $\eta\colon \mathrm{G}^0 \to \underline{\mathrm{Hom}}(\mathrm{A}, \Sigma^i\mathrm{F}^0)$ is null-homotopic with null-homotopy implemented by $\sigma_0\colon \mathrm{G}^0_0 \to \underline{\mathrm{Hom}}(\mathrm{A}, \Sigma^i\mathrm{F}^0)_1$, then σ_0 can be viewed as a natural transformation $\sigma_0\colon \mathrm{G} \to \underline{\mathrm{Hom}}'(\mathrm{A}_{i-1}, \mathrm{F})$. By the hom–tensor adjunction of Ab-functors, σ_0 corresponds to a natural transformation $\sigma'\colon \mathrm{G}\underline{\otimes}'\mathrm{A}_{i-1} \to \mathrm{F}$. Now observe that $(\mathrm{G}^0\underline{\otimes}\mathrm{A})_{i-1} = \mathrm{G}\underline{\otimes}'\mathrm{A}_{i-1}$ so $\sigma'\colon (\mathrm{G}^0\underline{\otimes}\mathrm{A})_{i-1} \to (\Sigma^i\mathrm{F}^0)_i$ implements a null-homotopy of the original natural transformation $\eta'\colon \mathrm{G}^0\underline{\otimes}\mathrm{A} \to \Sigma^i\mathrm{F}^0$.

The case $i = 0$: By the above calculation $\underline{\mathrm{Hom}}\big(\mathrm{A}, \mathrm{F}^0\big)(B)_0 \simeq \mathrm{Nat}\,\big(\mathrm{A}_0, \mathrm{F}(B\otimes_\sigma\_)\big)$, $\underline{\mathrm{Hom}}\big(\mathrm{A}, \mathrm{F}^0\big)(B)_{-1} \simeq \mathrm{Nat}\,\big(\mathrm{A}_1, \mathrm{F}(B\otimes_\sigma\_)\big)$ and the differential in degree 0 sends the natural transformation $\theta\colon \mathrm{Hom}_{\mathsf{A}}(A_0, \_) \to \mathrm{F}(B\otimes_\sigma\_)$ to $(-1)\theta \circ f^*\colon \mathrm{Hom}_{\mathsf{A}}(A_1, \_) \to \mathrm{F}(B\otimes_\sigma\_)$. Since η is a chain map, it follows that if $x \in \mathrm{G}(B)$ then $(-1)\big((\eta_B)_0(x)\big) \circ f^*$ is the zero natural transformation. By the earlier $i = 0$ case, this implies that $(\eta_B)_0(x)\colon \mathrm{A}_0 \to \mathrm{F}(B\otimes_\sigma\_)$ is the zero natural transformation. Since this holds for any object B of A and $x \in \mathrm{G}(B)$, $\eta\colon \mathrm{G}^0 \to \underline{\mathrm{Hom}}(\mathrm{A}, \mathrm{F}^0)$ must also be the zero natural transformation.

The case $i = 1$: In this case $\underline{\mathrm{Hom}}\big(\mathrm{A}, \Sigma\mathrm{F}^0\big)(B)_1 \simeq \mathrm{Nat}\,\big(\mathrm{A}_0, \mathrm{F}(B\otimes_\sigma\_)\big)$ and similarly $\underline{\mathrm{Hom}}\big(\mathrm{A}, \Sigma\mathrm{F}^0\big)(B)_0 \simeq \mathrm{Nat}\,\big(\mathrm{A}_1, \mathrm{F}(B\otimes_\sigma\_)\big)$. Moreover, the only non-zero differential is f^{**} in degree 1. By the earlier $i = 1$ case, if $x \in \mathrm{G}(B)$ then

$$(\eta_B)_0(x) = f^{**} \circ \mathrm{YF}_{A_0,B}^{-1} \circ \mathrm{F}(\mathrm{id}_B\otimes_\sigma f)^{-1} \circ \mathrm{YF}_{A_1,B}\big((\eta_B)_0(x)\big)$$

where $\mathrm{YF}_{A,B}\colon \mathrm{Nat}\,\big(\mathrm{Hom}_{\mathsf{A}}(A, \_), \mathrm{F}(B\otimes_\sigma\_)\big) \to \mathrm{F}(B\otimes_\sigma A)$, is the usual isomorphism from the Yoneda lemma. It remains to show that this is natural in B, that is that the collection $\{\mathrm{YF}_{A_0,B}^{-1} \circ \mathrm{F}(\mathrm{id}_B\otimes_\sigma f)^{-1} \circ \mathrm{YF}_{A_1,B} \circ (\eta_B)_0 \mid B \in \mathsf{A}\}$ gives a natural transformation $\mathrm{G}^0_0 \to \underline{\mathrm{Hom}}\big(\mathrm{A}, \Sigma\mathrm{F}^0\big)_1$. However, this readily follows from the naturality of the Yoneda isomorphism and the naturality of the isomorphism $\mathrm{F}(\mathrm{id}_B\otimes_\sigma f)$. $\square$

Homotopy invariance

The first obvious hurdle is that of homotopy invariance. It is desirable that homotopic maps $f_0, f_1 \colon A \to B$ between C*-algebras becomes the same morphism in the homotopy category. Recall that f_0 and f_1 are *homotopic* if there is a map $A \to B \otimes_\sigma \mathrm{C}(I)$ such that the leftmost part of diagram $(*)$ commutes,

$$\begin{tikzcd} & B & & B & \\ A \arrow[ur, bend left, "f_0"] \arrow[r] \arrow[dr, bend right, "f_1"'] & B\otimes_\sigma \mathrm{C}(I) \arrow[u, "ev_0"'] \arrow[d, "ev_1"] & & B\otimes_\sigma \mathrm{C}(I) \arrow[u, "ev_0"] \arrow[d, "ev_1"'] & B \arrow[l, "c_B"] \arrow[ul, bend right, "\mathrm{id}_B"'] \arrow[dl, bend left, "\mathrm{id}_B"] \\ & B & & B & \end{tikzcd} \tag{$*$}$$

while the rightmost part of the diagram $(*)$ is always commutative (here $c_B \colon B \to B\otimes_\sigma \mathrm{C}(I)$ is the map that sends b to the constant function $t \mapsto b$, while ev_t is evaluation at the point t). By considering the map $H \colon B\otimes_\sigma \mathrm{C}(I) \to B\otimes_\sigma \mathrm{C}(I)\otimes_\sigma \mathrm{C}(I) \simeq B\otimes_\sigma \mathrm{C}(I \times I)$ given by $H(b \otimes h) = b \otimes h_1 \otimes h_2$ with $(h_1 \otimes h_2)(t_1, t_2) = h(t_1 t_2)$ it is clear that $c_B \circ ev_0$ and $\mathrm{id}_{B\otimes_\sigma \mathrm{C}(I)}$ are homotopic.

Definition 2.2.3. A functor $\mathrm{F} \colon \mathsf{C^*\text{-}alg} \to \mathsf{C}$ is *homotopy invariant* if $\mathrm{F}(c_B)$ is an isomorphism for all objects B of $\mathsf{C^*\text{-}alg}$. ♠

From the discussion preceding Definition 2.2.3 it follows that a functor $\mathrm{F} \colon \mathsf{C^*\text{-}alg} \to \mathsf{C}$ is homotopy invariant if and only it takes homotopic maps to the same morphism in C.

Example 2.2.4. By [Hig87, Proposition 2.10], for any separable C*-algebra C, the functor $\mathrm{KK}(C, \_)$ is homotopy invariant. ♣

Consider the set

$$\mathscr{M}_H = \left\{ [c_A]^{\mathrm{Z}} \colon [A\otimes_\sigma \mathrm{C}(I)]^{\mathrm{Z}} \to [A]^{\mathrm{Z}} \;\middle|\; A \in \mathscr{A} \right\}$$

of morphisms in $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$, and let $\mathscr{C}_H$ be the set of (chain complex) mapping cones

$$\mathscr{C}_H = \left\{ \mathrm{cone}\left([c_A]^{\mathrm{Z}}\right) \;\middle|\; [c_A]^{\mathrm{Z}} \in \mathscr{M}_H \right\}.$$

From Proposition 1.4.11 one can left Bousfield localise with respect to the morphisms $0 \to \Sigma^n \mathrm{cone}\left([c_A]^{\mathrm{Z}}\right)$. Since the induced functor on homotopy categories is triangulated, this implies that the morphism $[c_A]^{\mathrm{Z}}$ is an isomorphism in the homotopy category of the localised model structure. However, in order for the localised model structure to be monoidal, one must involve the set $\mathcal{G}$.

Definition 2.2.5. Denote by $\mathscr{T}_H$ the set

$$\mathscr{T}_H = \left\{ \mathrm{cone}\left([c_A]^{\mathrm{Z}}\right) \underline{\otimes} \mathrm{G}^0 \;\middle|\; [c_A]^{\mathrm{Z}} \in \mathscr{M}_H, \mathrm{G} \in \mathcal{G} \right\}.$$

Bousfield localisation of the projective model structure on $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ with respect to the morphisms $0 \to \Sigma^n\big(\mathrm{cone}\,([c_A]^{\mathrm{Z}})\underline{\otimes}\mathrm{G}^0\big) \simeq \big(\Sigma^n\mathrm{cone}\,([c_A]^{\mathrm{Z}})\big)\underline{\otimes}\mathrm{G}^0$ where $[c_A]^{\mathrm{Z}}$ is in $\mathscr{M}_H$, $\mathrm{G} \in \mathcal{G}$ and $n \in \mathbf{Z}$. The collection of $\mathscr{T}_H$-local maps (i.e. the weak equivalences of the homotopy invariant model structure) will be denoted by $\mathscr{W}_H$. ♠

Note that since $\underline{\otimes}$ is an extension of $\otimes_\sigma$ and

$$\mathcal{G} = \{\mathrm{G} = \mathrm{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}}(G, \_)\colon \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0} \to \mathsf{Ab} \mid G \text{ is an element of } \mathscr{A}\}$$

it is clear that $\mathrm{cone}\,([c_A]^{\mathrm{Z}})\underline{\otimes}\mathrm{G}^0$ is isomorphic to $\mathrm{cone}\,([c_A\otimes_\sigma\mathrm{id}_G]^{\mathrm{Z}})$. Moreover $c_A\otimes_\sigma\mathrm{id}_G$ and $c_{A\otimes_\sigma G}$ only differs by a twist of the $\mathrm{C}(I)$ factor. So instead of $\mathscr{T}_H$ it is enough to consider the set $\{\mathrm{cone}\,([c_A]^{\mathrm{Z}}) \mid [c_A]^{\mathrm{Z}} \in \mathscr{M}_H\}$, whence only part 1 of Lemma 2.2.2 is needed. Note that one can do a similar process both for the coproduct property and matrix stability.

Proposition 2.2.6. *For any homotopy invariant functor* $\mathrm{F}\colon \mathsf{C^*\text{-}alg} \to \mathsf{Ab}$ *the object* $\mathrm{F}^0_{\mathsf{Ab}^0}$ *is* $\mathscr{T}_H$*-local.*

Proof. Let $C \in \mathsf{C^*\text{-}alg}$ and recall from Lemma 1.4.10 that $\mathrm{F}^0_{\mathsf{Ab}^0}$ is $\mathscr{T}_H$-local if for any T in $\mathscr{T}_H$ and $n \in \mathbf{Z}$ the group $\mathrm{Hom}_{D(\mathsf{Ab})}(T, \Sigma^n\mathrm{F}^0_{\mathsf{Ab}^0})$ is trivial. So let $\mathrm{cone}\,([c_A]^{\mathrm{Z}})$ be in $\mathscr{T}_H$ and note that both $[A]^{\mathrm{Z}}$ and $[A\otimes_\sigma\mathrm{C}(I)]^{\mathrm{Z}}$ are chain complexes of representable functors, and both are concentrated in degree 0.

Thus $\mathrm{cone}\,([c_A]^{\mathrm{Z}})$ is projective in each degree and bounded below whence Proposition 1.3.7 applies to $\mathrm{cone}\,([c_A]^{\mathrm{Z}})$, that is

$$\mathrm{Hom}_D\left(\mathrm{cone}\,([c_A]^{\mathrm{Z}}), \Sigma^n\mathrm{F}^0_{\mathsf{Ab}^0}\right) \simeq \mathrm{Hom}_K\left(\mathrm{cone}\,([c_A]^{\mathrm{Z}}), \Sigma^n\mathrm{F}^0_{\mathsf{Ab}^0}\right)$$

for all $n \in \mathbf{Z}$. Since the map $\mathrm{F}(c_A)$ is an isomorphism by assumption, Lemma 2.1.2 and Lemma 2.2.2 proves the result. □

Note that since the objects of $\mathscr{C}_H$ are cofibrant, so are the objects of $\mathscr{T}_H$. It follows that $\mathscr{T}_H$ is a flat family, whence the localised model structure is monoidal by Proposition 1.4.19.

Split exact extensions

An *extension of C*-algebras* is a diagram in $\mathsf{C^*\text{-}alg}$

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

with f injective, g surjective and $\mathrm{im}(f) = \ker(g)$. If there is a section $s\colon A'' \to A$ (so $g \circ s = \mathrm{id}_{A''}$), then the extension is said to be *split exact*. For commutative C*-algebras, this corresponds to the situation where $A'' = \mathrm{C}(X)$, $A = \mathrm{C}(Y)$ and X is a retraction of Y. One salient feature of KK-theory is that it has the coproduct property:

Definition 2.2.7. A functor $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{C}$ has the *coproduct property* if for any split extension

$$A' \xrightarrow{f} A \underset{s}{\overset{g}{\rightleftarrows}} A''$$

of C*-algebras, the maps $\mathrm{F}(f)$ and $\mathrm{F}(s)$ turn $\mathrm{F}(A)$ into the coproduct of $\mathrm{F}(A')$ and $\mathrm{F}(A'')$. ♠

Observe that if $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ has the coproduct property, then it takes split exact extensions to split exact sequences (in the sense of abelian groups). This explains why such functors are also called *split exact*. A proof of the split exactness of $\mathrm{KK}(C, \_)$ can be found in [Hig87, Proposition 2.12]. The next step is to archive a similar effect in $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Definition 2.2.8. Let $\mathscr{E}$ be the set of all sextuples (A, A', A'', f, g, s) such that

$$A' \xrightarrow{f} A \underset{s}{\overset{g}{\rightleftarrows}} A''$$

is a split extension of C*-algebras with $A, A', A'' \in \mathscr{A}$, and let $\mathscr{M}_{CP}$ be the set

$$\mathscr{M}_{CP} = \left\{ [f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\colon [A]^{\mathrm{Z}} \to [A']^{\mathrm{Z}} \oplus [A'']^{\mathrm{Z}} \;\middle|\; (A, A', A'', f, g, s) \in \mathscr{E} \right\}$$

of morphisms in $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$. Denote by $\mathscr{C}_{CP}$ the set of all (chain complex) mapping cones

$$\mathscr{C}_{CP} = \left\{ \mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right) \;\middle|\; [f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}} \in \mathscr{C}_{CP} \right\}$$

and by $\mathscr{T}_{CP}$ the set

$$\mathscr{T}_{CP} = \left\{ \mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right) \underline{\otimes} \mathrm{G}^0 \;\middle|\; \mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right) \in \mathscr{C}_{CP}, \mathrm{G} \in \mathcal{G} \right\}.$$

The *split exact* model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is the left Bousfield localisation of the projective model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ with respect to the morphisms $0 \to \Sigma^n \mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right) \underline{\otimes} \mathrm{G}^0$ where $[f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}$ is in $\mathscr{M}_{CP}$, $\mathrm{G} \in \mathcal{G}$ and $n \in \mathbf{Z}$. The collection of $\mathscr{T}_{CP}$-local maps (i.e. the weak equivalences of the split exact model structure) will be denoted by $\mathscr{W}_{CP}$. ♠

Similar to the homotopy invariant situation, all of the objects of $\mathscr{C}_{CP}$ are cofibrant, so $\mathscr{T}_{CP}$ is a flat family. Thus the localised model structure is monoidal by Proposition 1.4.19.

Proposition 2.2.9. *If* $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ *has the coproduct property then the object* $\mathrm{F}^0_{\mathsf{Ab}^0}$ *is* $\mathscr{T}_{CP}$*-local.*

Proof. The idea of the proof is the same as that of Proposition 2.2.6. Fix an element $\mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right) \in \mathscr{T}_{CP}$ and note that both $[A]^{\mathrm{Z}}$ and $[A']^{\mathrm{Z}} \oplus [A'']^{\mathrm{Z}}$ are chain complexes of representable functors, and both are concentrated in degree 0. So $\mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right)$

is projective in each degree and bounded below whence Proposition 1.3.7 is applicable. Thus

$$\mathrm{Hom}_D\left(\mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right), \Sigma^n \mathrm{F}^0_{\mathsf{Ab}^0}\right) \simeq \mathrm{Hom}_K\left(\mathrm{cone}\left([f]^{\mathrm{Z}} \times [s]^{\mathrm{Z}}\right), \Sigma^n \mathrm{F}^0_{\mathsf{Ab}^0}\right)$$

for all $n \in \mathbf{Z}$. By assumption the map $\mathrm{F}(f) + \mathrm{F}(s)\colon \mathrm{F}(A') \oplus \mathrm{F}(A'') \to \mathrm{F}(A)$ given by $(a', a'') \mapsto \mathrm{F}(f)(a') + \mathrm{F}(s)(a'')$ is an isomorphism so it remains to modify Lemma 2.2.2 to work in this case. The first part of this modification is the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A', _), \mathrm{F}_{\mathsf{Ab}^0}\right) \oplus \mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A'', _), \mathrm{F}_{\mathsf{Ab}^0}\right) \arrow[r, "\mathrm{YF}_{A'} \oplus \mathrm{YF}_{A''}"] \arrow[d, "\simeq"', "S"] & \mathrm{F}(A') \oplus \mathrm{F}(A'') \arrow[dd, "\mathrm{F}(f)+\mathrm{F}(s)"] \\
\mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A', _) \oplus \mathrm{Hom}_{\mathsf{A}}(A'', _), \mathrm{F}_{\mathsf{Ab}^0}\right) \arrow[d, "(f^* \times s^*)^*"] & \\
\mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{A}}(A, _), \mathrm{F}_{\mathsf{Ab}^0}\right) \arrow[r, "\mathrm{YF}_A"] & \mathrm{F}(A)
\end{tikzcd}$$

where $\mathsf{A} = \mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ and S is the isomorphism $S(\eta', \eta'')_B(g', g'') = \eta'_B(g') + \eta''_B(g'')$. This follows by a computation similar to the one in the proof of Lemma 2.2.1. The second part of the modification is to find a lifting in the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{A}}(A, _) \arrow[r, "\eta_1"] \arrow[d, "f^* \times s^*"] & \mathrm{F}_{\mathsf{Ab}^0} \arrow[d] \\
\mathrm{Hom}_{\mathsf{A}}(A', _) \oplus \mathrm{Hom}_{\mathsf{A}}(A'', _) \arrow[r] & 0,
\end{tikzcd}$$

and in this case the natural transformation $\sigma_0 = S \circ \left(\mathrm{YF}_{A'} \oplus \mathrm{YF}_{A''}\right)^{-1} \circ \left(\mathrm{F}(f) + \mathrm{F}(s)\right)^{-1} \circ \mathrm{YF}_A(\eta_1)$ does the trick. □

Matrix stability

One important aspect of KK-theory is matrix stability. Let $\mathbb{K}$ be the compact operators on the Hilbert space ℓ^2, and note that one can include the C*-algebra of $n \times n$-matrices over A into $A \otimes_\sigma \mathbb{K}$. Now fix an one dimensional projection $p \in \mathbb{K}$ and for a C*-algebra A define the $*$-preserving algebra morphism $k_A\colon A \to A \otimes_\sigma \mathbb{K}$ by $a \mapsto a \otimes p$ (note that two different choices of projection p gives homotopic morphisms of C*-algebras). The matrix stability of KK-theory is that the morphism k_A becomes an isomorphism in KK. Thus the functor $\mathrm{KK}(C, \_)$ is an example of a matrix stable functor [Hig87, Proposition 2.11].

Definition 2.2.10. A functor $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{C}$ is *matrix stable* if $\mathrm{F}(k_A)$ is an isomorphism for all objects A of $\mathsf{C}^*\text{-}\mathsf{alg}$. ♠

Definition 2.2.11. Let $\mathscr{M}_S$ be the set of morphisms

$$\mathscr{M}_S = \left\{ [k_A]^{\mathrm{Z}}\colon [A \otimes_\sigma \mathbb{K}]^{\mathrm{Z}} \to [A]^{\mathrm{Z}} \;\middle|\; A \in \mathscr{A} \right\},$$

$\mathscr{C}_S$ the set of (chain complex) mapping cones

$$\mathscr{C}_S = \{\text{cone}\,([k_A]^{\mathrm{Z}}) \mid [k_A]^{\mathrm{Z}} \in \mathscr{C}_S\},$$

and $\mathscr{T}_S$ the set

$$\mathscr{T}_S = \left\{\text{cone}\,([k_A]^{\mathrm{Z}})\underline{\otimes}\mathrm{G}^0 \;\middle|\; [k_A]^{\mathrm{Z}} \in \mathscr{M}_S, \mathrm{G} \in \mathcal{G}\right\}.$$

The *matrix stable* model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is the left Bousfield localisation of the projective model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ with respect to the morphisms $0 \to \Sigma^n\text{cone}\,([k_A]^{\mathrm{Z}})\underline{\otimes}\mathrm{G}^0$ where $[k_A]^{\mathrm{Z}}$ is in $\mathscr{M}_S$, $\mathrm{G} \in \mathcal{G}$ and $n \in \mathbf{Z}$. The collection of $\mathscr{T}_S$-local maps (i.e. the weak equivalences of the matrix stable model structure) will be denoted by $\mathscr{W}_S$. ♠

The matrix stable model structure is monoidal by Proposition 1.4.19 and the fact that all of the objects of $\mathscr{C}_S$ are cofibrant.

Proposition 2.2.12. *For any matrix stable functor* $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ *the object* $\mathrm{F}^0_{\mathsf{Ab}^0}$ *is* $\mathscr{T}_S$*-local.*

Proof. This follows by the same procedure as the proof of Proposition 2.2.6. □

Relationship with KK-theory

The localisation machinery of above was motivated by KK-theory. Thus one could hope that the category $\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}))$ is related to the category KK. This is indeed the case, as there is a particular subcategory of the "opposite" homotopy category $\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}))^{\mathrm{op}}$ which is isomorphic to KK.

The above mentioned subcategory is basically the localisation of $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ by the weak equivalences. With this in mind, denote by $\mathscr{W}'_H$ the collection of morphisms $f\colon A \to B$ in $\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}$ such that $\mathrm{Y}(f)^0\colon [B]^{\mathrm{Z}} \to [A]^{\mathrm{Z}}$ is in $\mathscr{W}_H$ (where $\mathrm{Y}\colon \mathsf{C}^*\text{-}\mathsf{alg}^{\mathrm{op}}_{\mathsf{Ab}^0} \to [\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}$ is the Ab-enriched contravariant Yoneda embedding), and define the collections $\mathscr{W}'_{CP}$ and $\mathscr{W}'_S$ similarly.

Proposition 2.2.13. *If* $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ *is homotopy invariant then* $\mathrm{F}_{\mathsf{Ab}^0}\colon \mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0} \to \mathsf{Ab}$ *takes any morphism in* $\mathscr{W}'_H$ *to an isomorphism. Similarly, if* $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ *has the coproduct property (resp. is matrix stable) then* $\mathrm{F}_{\mathsf{Ab}^0}\colon \mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0} \to \mathsf{Ab}$ *takes any morphism in* $\mathscr{W}'_{CP}$ *(resp.* $\mathscr{W}'_S$*) to an isomorphism.*

Proof. Since the proofs involving the coproduct property and matrix stability are almost identical to the one for homotopy invariance, only the latter will be proved. Assume $\mathrm{F}\colon \mathsf{C}^*\text{-}\mathsf{alg} \to \mathsf{Ab}$ is homotopy invariant and $f\colon A \to B$ is a morphism in $\mathscr{W}'_H$. Since the object $\mathrm{F}^0_{\mathsf{Ab}^0}$ is $\mathscr{T}_H$-local by Proposition 2.2.6, the morphism $\mathrm{Y}(f)^0\colon [B]^{\mathrm{Z}} \to [A]^{\mathrm{Z}}$ induces a bijection $\mathrm{Hom}_{D([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})}([A]^{\mathrm{Z}}, \mathrm{F}^0_{\mathsf{Ab}^0}) \to \mathrm{Hom}_{D([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})}([B]^{\mathrm{Z}}, \mathrm{F}^0_{\mathsf{Ab}^0})$. By Proposition 1.3.7, the fact that $[A]^{\mathrm{Z}}$, $[B]^{\mathrm{Z}}$ and $\mathrm{F}^0_{\mathsf{Ab}^0}$ are all concentrated in degree 0 and the Yoneda lemma, this bijection gives an isomorphism $\mathrm{F}(f)\colon \mathrm{F}(A) \to \mathrm{F}(B)$. □

By Proposition 1.4.11 the category $\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))$ is additive. Moreover, the functor $\Upsilon(\_)^0\colon \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}^{\mathrm{op}} \to \mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}})$ induces a functor

$$\Upsilon\colon \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}[(\mathscr{W}_H'\cup\mathscr{W}_{CP}'\cup\mathscr{W}_S')^{-1}]^{\mathrm{op}} \to \mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))$$

by the definitions of $\mathscr{W}_H'$, $\mathscr{W}_{CP}'$ and $\mathscr{W}_S'$.

Definition 2.2.14. Denote by kk the category with the same objects as $\mathsf{C^*\text{-}alg}$, but where $\mathrm{Hom}_{\mathsf{kk}}(A,B)$ is the subgroup of $\mathrm{Hom}_{\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))}([B]^{\mathrm{Z}},[A]^{\mathrm{Z}})$ generated by the elements of the form $\Upsilon(f)$ for $f\in\mathrm{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}[(\mathscr{W}_H'\cup\mathscr{W}_{CP}'\cup\mathscr{W}_S')^{-1}]}(A,B)$. Define the functor $\mathrm{kk}\colon \mathsf{C^*\text{-}alg}\to\mathsf{kk}$ to be the composite

$$\begin{tikzcd}
\mathsf{C^*\text{-}alg} \arrow[r] & \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0} \arrow[r] & \mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}[(\mathscr{W}_H'\cup\mathscr{W}_{CP}'\cup\mathscr{W}_S')^{-1}] \arrow[d, "\Upsilon^{\mathrm{op}}"] \\
 & & \mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))^{\mathrm{op}}
\end{tikzcd}$$

whose image clearly lie in kk. ♠

Proposition 2.2.15. *The category* kk *is canonically isomorphic to* KK*. Under this isomorphism the functors* kk *and* KK *agree.*

Proof. Note that any morphism $f\colon A\to B$ in kk is of the form $\sum_{i=1}^n \alpha_i f_i$ where $\alpha_i\in\mathbf{Z}$ and f_i is an equivalence class of strings $f_{i,1}w_{i,1}^{-1}\cdots f_{i,n_i}w_{i,n_i}^{-1}$ with $f_{i,j}$ a morphism of $\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$ and $w_{i,j}$ a morphism in $\mathscr{W}_H'\cup\mathscr{W}_{CP}'\cup\mathscr{W}_S'$. The equivalence relation is generated by saying that the empty string, the string ww^{-1} and the string $w^{-1}w$ are all equivalent to the identity.

Let $\mathrm{F}\colon \mathsf{C^*\text{-}alg}\to\mathsf{Ab}$ be a homotopy invariant and matrix stable functor with the coproduct property. Define $\mathrm{F}''\colon \mathsf{kk}\to\mathsf{Ab}$ by requiring that

$$\mathrm{F}''\Big(\sum_{i=1}^n \alpha_i f_{i,1}w_{i,1}^{-1}\cdots f_{i,n_i}w_{i,n_i}^{-1}\Big) = \sum_{i=1}^n \alpha_i \mathrm{F}(f_{i,1})\circ\mathrm{F}(w_{i,1})^{-1}\circ\cdots\circ\mathrm{F}(f_{i,n_i})\circ\mathrm{F}(w_{i,n_i})^{-1}$$

and note that $\mathrm{F}''\circ\mathrm{kk}=\mathrm{F}$. This is well defined by Proposition 2.2.13, and since the category $\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))$ is additive, so is kk. Thus the pair $(\mathsf{kk},\mathrm{kk})$ has the same universal property as the pair $(\mathsf{KK},\mathrm{KK})$ from Proposition 2.1.7. ☐

Corollary 2.2.16. *The group* $\mathrm{Hom}_{\mathsf{Ho}_{\mathscr{W}_H\cup\mathscr{W}_{CP}\cup\mathscr{W}_S}(\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}}))}([B]^{\mathrm{Z}},[A]^{\mathrm{Z}})$ *contains* $\mathrm{KK}(A,B)$ *as a subgroup.* ☐

The G-equivariant case

In this subsection G will be a fixed locally compact group* which is also σ-compact†. Recall that a C*-algebra A is a G-C*-algebra if there is an action of G on A, that is

* I.e. G is a locally compact Hausdorff topological space, and both the multiplication map and the inverse map are continuous. † I.e. G can be written as a countable union of compact subspaces.

a group homomorphism : $G \to \mathrm{Aut}(A)$ such that the map : $G \to A$ is continuous for all $a \in A$. If A and B are G-C*-algebras then a C*-algebra morphism $f\colon A \to B$ is G-equivariant if it is compatible with the action of G.

The G-equivariant version of KK-theory was introduced by Kasparov in his 1981 conspectus[Kas95]. Similar to the non-equivariant case, this theory leads to an additive category KK^G. The universal property satisfied by this category was provided by Klaus Thomsen in [Tho98].

Two G-equivariant morphism $f_0, f_1\colon A \to B$ are *homotopic* if for each $a \in A$ there is a continuous map $F_a\colon I \to B$ such that $F_a(0) = f_0(a)$, $F_a(1) = f_1(a)$, and for each $t \in I$ the map $f_t\colon A \to B$ given by $f_t(b) = F_t(b)$ is a G-equivariant C*-algebra morphism. Note that two morphisms are homotopic if and only if there is a G-equivariant morphism $F\colon A \to B \otimes_\sigma \mathrm{C}(I)$ such that $ev_0 \circ F = f_0$, $ev_1 \circ F = f_1$ and G acts only on the B part of $B \otimes_\sigma \mathrm{C}(I)$.

An *extension of G-C*-algebras* is an extension of C*-algebras where all the maps are G-equivariant. It is a split-exact extension if it is split-exact as an extension of C*-algebras with a G-equivariant section.

Denote by G-$\mathsf{C^*\text{-}alg}$ the category of G-C*-algebras. A functor $\mathrm{F}\colon G\text{-}\mathsf{C^*\text{-}alg} \to \mathsf{C}$ is *homotopy invariant* if it takes homotopic maps to the same morphism, *matrix stable* if it takes $k_A\colon A \to A \otimes_\sigma \mathbb{K}$ to an isomorphism* and has the *coproduct property* if for any split extension

$$A' \xrightarrow{f} A \underset{s}{\overset{g}{\rightleftarrows}} A''$$

of G-C*-algebras, the maps $\mathrm{F}(f)$ and $\mathrm{F}(s)$ turn $\mathrm{F}(A)$ into the coproduct of $\mathrm{F}(A')$ and $\mathrm{F}(A'')$.

Proposition 2.2.17 ([Tho98, Theorem 2.2])**.** *Any functor* $\mathrm{F}\colon G\text{-}\mathsf{C^*\text{-}alg} \to \mathsf{Ab}$ *which is homotopy invariant, matrix stable and has the coproduct property factors through the category* KK^G.

Due to the similarity of the axioms for G-equivariant KK-theory and ordinary KK-theory, the obvious modification of Proposition 2.2.15 holds true in this case.

The problem with E-theory and $\Sigma\mathrm{Ho}$

Consider an extension

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

of C*-algebras with a *contractive completely positive linear section* $s\colon A'' \to A$. Such extensions are called *cpc-split*† and in this case the corresponding sequence in KK is exact in the middle. This is the approach to KK-theory used in [Cun98] since $\mathrm{KK}\colon \mathsf{C^*\text{-}alg} \to \mathsf{KK}$ is the universal functor satisfying homotopy invariance and matrix stability which is also half-exact for cpc-split extensions‡ ([Cun98, Proposition 3.1]). Similarly, the functor

* Here G only acts on the A part of $A \otimes_\sigma \mathbb{K}$. It follows that the map k_A is G-equivariant. † Where "cpc" is short for *completely positive contractive*. Note that a contractive section s must have norm 1 since g is contractive. ‡ I.e. it takes cpc-split extensions to sequences which are exact in the middle. Note that in [Cun98] the terminology is that such functors "satisfy excision on cpc-split extensions".

$\mathsf{C^*\text{-}alg} \to \Sigma\mathrm{Ho}$ from [CMR07] is the universal functor satisfying homotopy invariance which is half-exact for cpc-split extensions.

A related approach is E-theory, which was introduced by Alain Connes and Nigel Higson in [CH90] using asymptotic morphisms. This theory leads to an additive category E and a functor $\mathrm{E}\colon \mathsf{C^*\text{-}alg} \to \mathsf{E}$ which is the universal functor satisfying homotopy invariance and matrix stability which is half-exact for all extensions[*] ([Hig90, Remark 3.3]).

It is possible to use the localisation machinery to attempt to model both E-theory and the category $\Sigma\mathrm{Ho}$. For E-theory, consider the set $\mathscr{E}'$ of all quintuples (A, A', A'', f, g) such that

$$A' \xrightarrow{f} A \xrightarrow{g} A''$$

is an extension of C*-algebras with $A, A', A'' \in \mathscr{A}$, and let $\mathscr{M}_E$ be the set

$$\mathscr{M}_E = \left\{ \operatorname{im}[g]^{\mathrm{Z}} \to \ker[f]^{\mathrm{Z}} \;\middle|\; (A, A', A'', f, g) \in \mathscr{E}' \right\}$$

of morphisms in $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$. One can then form the left Bousfield localisation of the projective model structure on $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ with respect to the morphisms $0 \to \Sigma^n \operatorname{cone}(e) \underline{\otimes} \mathrm{G}^0$ where e is in $\mathscr{M}_E$, $\mathrm{G} \in \mathcal{G}$ and $n \in \mathbf{Z}$. However, in order to use Lemma 2.2.2, the functors in the chain complex must be representable[†]. Denote by $\operatorname{im} f$ the category theoretic image of f, i.e. the closed ideal generated by the set theoretic image of f. The C*-algebra $\operatorname{coker} f = A/\operatorname{im} f$ is the most likely candidate for the representing object of $\ker \mathrm{Y}(f)$. However, this fails as $\operatorname{coker} f$ is not necessarily the cokernel of f in $\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$. To see this let $f\colon \mathbf{C} \to \mathbf{C} \oplus \mathbf{C}$ be the map $x \mapsto (x, 0)$ and $h\colon \mathbf{C} \oplus \mathbf{C} \to \mathbf{C} \oplus \mathbf{C}$ be the map $(x, y) \mapsto (x, 0)$. In this case $\operatorname{coker} f \simeq \mathbf{C}$ and $h \circ f = \mathrm{id}_{\mathbf{C}\oplus\mathbf{C}} \circ f$. It follows that the morphisms $h - \mathrm{id}_{\mathbf{C}\oplus\mathbf{C}}$ factors through the cokernel of f in $\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$. However, $\mathrm{id}_{\mathbf{C}\oplus\mathbf{C}}$ does not factor through $\mathbf{C}$, so $\operatorname{coker} f$ can not be the cokernel of f in $\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}$.

A similar problem arises when one tries to obtain the category $\Sigma\mathrm{Ho}$ by localisation. In this case let $\mathscr{M}_{CPC} = \{\operatorname{im}[g]^{\mathrm{Z}} \to \ker[f]^{\mathrm{Z}} \mid (A, A', A'', f, g, s) \in \mathscr{E}''\}$ where $\mathscr{E}''$ is the set of all sextuples (A, A', A'', f, g, s) such that

$$\begin{tikzcd} A' \arrow[r, "f"] & A \arrow[r, "g"] & A'' \arrow[l, bend right, "s"'] \end{tikzcd}$$

is a cpc-split extension of C*-algebras with $A, A', A'' \in \mathscr{A}$. The Bousfield localisation will then be on the set of morphisms $0 \to \Sigma^n \operatorname{cone}(e) \underline{\otimes} \mathrm{G}^0$ with $e \in \mathscr{M}_{CPC} = \{\operatorname{im}[g]^{\mathrm{Z}} \to \ker[f]^{\mathrm{Z}} \mid (A, A', A'', f, g, s) \in \mathscr{E}''\}$, and this give the same problem as the ones encountered in E-theory.

2.3 Simplicial C*-spaces

In [Øst10] Østvær studies cubical C*-spaces, and uses those to form a homotopy theory for C*-algebras. A similar setup can also be done for simplicial C*-spaces. The aim

[*] I.e. E takes all extensions of C*-algebras to sequences which are exact in the middle. [†] In view of the proof of Proposition 2.2.9 this requirement can be weakened somewhat.

of this section is to give a brief outline of simplicial C*-spaces and use the Dold–Kan correspondence to compare simplicial C*-spaces to $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Definition 2.3.1. A *C*-space* is a functor from C*-algebras to sets. The category $\mathsf{C^*\text{-}spc}$ has C*-spaces as objects and natural transformations as morphisms. If the sets have basepoints, then one speak of the category of pointed C*-spaces, $\mathsf{C^*\text{-}spc}_*$.

A *simplicial C*-space* is a functor from C*-algebras to simplicial sets. Similar to the above, the category $\Delta\mathsf{C^*\text{-}spc}$ has simplicial C*-spaces as objects and natural transformations as morphisms. There is also the pointed version $\Delta\mathsf{C^*\text{-}spc}_*$. ♠

A classical example of an adjunction is the hom–tensor adjunction of a monoidal category C. A special case of this adjunction is when the category C is Cartesian closed, that is it has finite (categorical) products, and this product coincides with the monoidal product. Note that all Cartesian closed categories are symmetric monoidal. A particular example of this is the category of small categories (with functors as morphisms), and in this case the adjunction gives an isomorphism of categories. Thus the category $\Delta\mathsf{C^*\text{-}spc}$ could equally well be described as the category of functors from Δ^{op} to C*-spaces (i.e. the category $s\mathsf{C^*\text{-}alg}$ of simplicial C*-algebras). By the same method $\Delta\mathsf{C^*\text{-}spc}$ is also equivalent to the category of functors from $\Delta^{\mathrm{op}} \times \mathsf{C^*\text{-}alg}$ to Set.

Another archetypical example is the "free–forgetful" adjunction. More precisely, let $\mathbf{F}\colon \mathsf{Set} \to \mathsf{Ab}$ be the functor that takes a set to the free abelian group with generators this set and let $\mathbf{U}\colon \mathsf{Ab} \to \mathsf{Set}$ be the forgetful functor. These functors are adjoint, that is if G is an abelian group and X is a set, then there is a natural bijection $\mathrm{Hom}_{\mathsf{Ab}}\big(\mathbf{F}(X), G\big) \to \mathrm{Hom}_{\mathsf{Set}}\big(X, \mathbf{U}(G)\big)$. Note that this adjunction can be extended to an adjunction between functor categories

$$\mathbf{F}_*\colon [\mathsf{C}, \mathsf{Set}] \rightleftarrows [\mathsf{C}, \mathsf{Ab}]\colon \mathbf{U}_*$$

for any category C. There is a similar adjunction in the pointed setting.

Consider the diagram ($*$) below. In this diagram, the "free–forgetful" adjunction forms the basis of the rightmost part. On the other hand, a slight modification of Proposition 1.2.8 yields an isomorphism

$$\mathsf{Ch}^+([\mathsf{C}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}}) \to [\mathsf{C}_{\mathsf{Ab}}, \mathsf{Ch}^+(\mathsf{Ab})]_{\mathsf{Ab}}$$

where the + superscript on Ch indicates that it is the category of connective chain complexes*. Combining this with Lemma 2.1.3, one obtain the leftmost part of the diagram

$$\begin{tikzcd}
\mathsf{Ch}^+([\mathsf{C}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}}) \arrow[d, Leftrightarrow] & & \\
{[\mathsf{C}_{\mathsf{Ab}}, \mathsf{Ch}^+(\mathsf{Ab})]_{\mathsf{Ab}}} \arrow[d, Leftrightarrow] & & \\
{[\mathsf{C}, \mathsf{Ch}^+(\mathsf{Ab})]} \arrow[r, leftrightarrow, dotted] & {[\mathsf{C}, [\Delta^{\mathrm{op}}, \mathsf{Ab}]]} \arrow[r, shift right, leftarrow] \arrow[r, shift left] & {[\mathsf{C}, [\Delta^{\mathrm{op}}, \mathsf{Set}]].}
\end{tikzcd} \tag{$*$}$$

* A chain complexes is connective if it is zero in all negative degrees.

The remaining part of the diagram $(*)$ is the dotted arrow, which stems from the Dold–Kan correspondence. One way to state this correspondence is to say that the normalised chain complex functor gives an equivalence of categories [Dol58, Theorem 1.9].

Definition 2.3.2. Let A be an abelian category. The *normalised chain complex functor* $\mathbf{N}\colon s\mathsf{A} = [\Delta^{\mathrm{op}}, \mathsf{A}] \to \mathsf{Ch}^+(\mathsf{A})$ takes a simplicial object $X_\bullet$ in A to its normalised Moore complex, that is the chain complex which in degree n consist of the non degenerate n-simplicies of $X_\bullet$. Explicitly, given a simplicial object $X_\bullet$ in A with face maps $d_i^n\colon X_n \to X_{n-1}$ for $0 \leq i \leq n$ and $n > 0$, its normalised Moore complex is $\big(\mathbf{N}(X_\bullet)_n, \partial_n\big)$ where

$$\mathbf{N}(X_\bullet)_n = \begin{cases} \bigcap_{i=0}^{n-1} \ker d_i^n & n > 0, \\ X_0 & n = 0, \\ 0 & n < 0, \end{cases}$$

and $\partial_n = (-1)^n d_n^n$ for $n > 0$. The effect of $\mathbf{N}$ on a simplicial group homomorphism $f_\bullet\colon X_\bullet \to Y_\bullet$ is the chain map $\mathbf{N}(f_\bullet)$ where $\mathbf{N}(f_\bullet)_n$ the restriction of f_n to $\mathbf{N}(X)_n$. ♠

In order for $\mathbf{N}$ to be part of an equivalence of categories, there must be a corresponding functor $\mathbf{L}\colon \mathsf{Ch}^+(\mathsf{Ab}) \to s\mathsf{Ab}$. Such a functor is due to Daniel Kan ([Dol58, Definition 1.8]). For convenience, the definition is repeated in Definition 2.3.3.

Definition 2.3.3. In the category Δ, let $\sigma_i^{n+1}\colon \mathbf{n+1} \to \mathbf{n}$ be the ith codegeneracy map and $\delta_i^{n-1}\colon \mathbf{n-1} \to \mathbf{n}$ be the ith coface map*. Denote by N^n the chain complex $\mathbf{NF}_*\Delta^n = \mathbf{NF}_* \operatorname{Hom}_\Delta(\_, \mathbf{n})$, and let

$$\epsilon_i^n = \mathbf{NF}_*\big((\delta_i^{n-1})_*\big)\colon N^{n-1} \to N^n,$$
$$\eta_i^n = \mathbf{NF}_*\big((\sigma_i^{n+1})_*\big)\colon N^{n+1} \to N^n.$$

The functor $\mathbf{L}\colon \mathsf{Ch}^+(\mathsf{Ab}) \to s\mathsf{Ab}$ sends a connective chain complex $C = (C_k, \partial_k)$ to the simplicial abelian group $X_\bullet$ where $X_n = \operatorname{Hom}_{\mathsf{Ch}^+(\mathsf{Ab})}(N^n, C)$, the face maps are

$$d_i^n = (\epsilon_i^n)^*\colon X_n \to X_{n-1},$$

and the degeneracy maps are

$$s_i^n = (\eta_i^n)^*\colon X_n \to X_{n+1}.$$ ♠

Remark 2.3.4. It is possible to give another definition of the X_k appearing in Definition 2.3.3, namely $X_n \simeq \bigoplus_{f \in S(\mathbf{n})} C_{\mathrm{cod} f}$ where $S(\mathbf{n}) = \{f\colon \mathbf{n} \to \mathbf{k} \mid f \text{ is surjective}\}$ and cod takes $f\colon \mathbf{n} \to \mathbf{k}$ to k. For $f\colon \mathbf{n} \to \mathbf{k}$ a surjective map, denote by $\chi_f^n\colon C_f = C_{\mathrm{cod} f} \to X_n$ the inclusion. The degeneracy maps of $X_\bullet$ are then given by $s_i^n|_{C_f} = \chi_{f \circ \sigma_i^n}^{n+1}$, and it is

* So for a simplicial complex X its degeneracy maps are $s_i^{n+1} = (\sigma_i^{n+1})^*\colon X_n \to X_{n+1}$, while its face maps are $d_i^{n-1} = (\delta_i^{n-1})^*\colon X_n \to X_{n-1}$.

also possible to describe the face maps using the inclusions. Of particular interest are the following:

$$d_i^n|_{C_{\mathrm{id}_n}} = \begin{cases} 0 & i \neq n, \\ \chi^{n-1}_{\mathrm{id}_{n-1}} \circ \partial_n & i = n, \text{ and} \end{cases}$$

$$d_i^n|_{C_{\sigma_j^n}} = \begin{cases} \chi^{n-1}_{\mathrm{id}_{n-1}} & i = j \text{ or } i = j+1, \\ \chi^{n-1}_{\sigma_j^{n-1}} \circ \partial_{n-1} & i = n \text{ and } j < n-1, \\ 0 & \text{otherwise.} \end{cases}$$

For more details, see [Wei95, 8.4.4]. ■

One reason to study simplicial sets is that they are a model for homotopy theory of spaces. More precisely, there is a model structure on $s\mathsf{Set} = [\Delta^{\mathrm{op}}, \mathsf{Set}]$ (the Quillen model structure) so that $\mathsf{Ho}(s\mathsf{Set}) \simeq \mathsf{Ho}(\mathsf{Top})$ (for a sufficiently nice* category Top of spaces).

The Quillen model structure on simplicial set forms the basis for model structures on simplicial C*-spaces, and this forms the foundation of Østvær's homotopy theory for C*-algebras[Øst10]. In Proposition 2.3.9 the projective model structures on C*-spaces is compared to corresponding model structures on $\mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Recall that there is a functor $|\_|\colon s\mathsf{Set} \to \mathsf{Top}$ called *geometric realization* (for details see [GJ09, pp. 7–8]) and that a morphism $f\colon X \to Y$ between simplicial sets is a *weak homotopy equivalence* if the map $|f|\colon |X| \to |Y|$ is a weak homotopy equivalence†.

Definition 2.3.5. For $n \geq 0$, the *simplicial n-simplex* is the simplicial set $\Delta^n_\bullet = \mathrm{Hom}_\Delta(\_, \mathbf{n})$ (i.e. $\Delta^n_l = \mathrm{Hom}_\Delta(\mathbf{l}, \mathbf{n})$). Denote its degeneracy maps by $s^l_j\colon \Delta^n_l \to \Delta^n_{l+1}$. for $0 \leq j \leq l$.

The *simplicial boundary* $(n-1)$*-sphere* is the simplicial set $\partial\Delta^n_\bullet$ with

$$\partial\Delta^n_l = \begin{cases} \Delta^n_l & l \leq n-1, \\ \left\{ s^{l-1}_j(\alpha) \;\middle|\; \alpha \in \partial\Delta^n_{l-1}, 0 \leq j \leq l-1 \right\} & l > n-1, \end{cases}$$

and whose face and degeneracy maps are restriction of the corresponding face and degeneracy maps of $\Delta^n_\bullet$.

Fix integers i and n with $0 \leq i \leq n$ and denote by $\kappa^{n,i}\colon \mathbf{n-1} \to \mathbf{n}$ the morphism in Δ given by

$$\kappa^{n,i}(k) = \begin{cases} k & k < i, \\ k+1 & k \geq i. \end{cases}$$

* E.g. where all the spaces have homotopy type of a CW-complex. † A map of topological spaces is a weak homotopy equivalence if it induces an isomorphism on all homotopy groups and a bijection on π_0. Note that by Whitehead's theorem a weak homotopy equivalence between CW-complexes is a homotopy equivalence[Hat10, Theorem 4.5].

The *ith horn of the simplicial n-simplex* is the simplicial set $\Lambda^{n,i}_\bullet$ with

$$\Lambda^{n,i}_l = \begin{cases} \Delta^n_l & l < n-1, \\ \Delta^n_l \setminus \{\kappa^{n,i}\} & l = n-1, \\ \left\{ s^{l-1}_j(\alpha) \;\middle|\; \alpha \in \Lambda^{n,i}_{l-1}, 0 \leq j \leq l-1 \right\} & l > n-1, \end{cases}$$

and whose face and degeneracy maps are the restriction of the corresponding face and degeneracy maps of $\Delta^n_\bullet$. ♠

Note that there is an obvious inclusion of $\partial\Delta^n_\bullet$ into $\Delta^n_\bullet$ and of $\Lambda^{n,i}_\bullet$ into $\Delta^n_\bullet$. These inclusions combined with the weak homotopy equivalences provides the language to define the Quillen model structure on $s\mathsf{Set}$.

Definition 2.3.6. Let I be the set of inclusions $\partial\Delta^n_\bullet \to \Delta^n_\bullet$ and J be the set of inclusions $\Lambda^{n,i}_\bullet$ into $\Delta^n_\bullet$. The *Quillen model structure on* $s\mathsf{Set}$ is cofibrantly generated with cofibrations the I-cofibrations, fibrations the J-injective and weak equivalences the weak homotopy equivalences*.

In the pointed setting the Quillen model structure on $s\mathsf{Set}_*$ is obtained analogous – the only change is that disjoint basepoints are added to all domains and codomains of morphisms in I and J. ♠

Observe that a morphism in the Quillen model structure on $s\mathsf{Set}$ is a fibration if and only if it is J-injective, that is it has the right lifting property with respect to all inclusions of horns. Thus the fibrations are precisely the *Kan-fibrations*. Moreover, one can show that a morphism is a cofibration if and only if it is a monomorphism ([GJ09, p. 62] or [Hov99, Proposition 3.2.2]), whence all objects are cofibrant.

Since the image of the Yoneda embedding of Δ is a small and dense subcategory of $s\mathsf{Set}$, the model structure is combinatorial by Vopěnka's principle (Proposition A.10). Moreover, by [Hir03, Theorem 13.1.13] the Quillen model structure is proper.

The Quillen model structure on $s\mathsf{Set}$ combined with the forgetful functor $\mathbf{U}\colon \mathsf{Ab} \to \mathsf{Set}$ gives rise to the Quillen projective model structure on $s\mathsf{Ab} = [\Delta^{\mathrm{op}}, \mathsf{Ab}]$.

Definition 2.3.7. Suppose $f_\bullet\colon X_\bullet \to X'_\bullet$ is a morphism of simplicial abelian groups. In the *Quillen projective model structure* on $s\mathsf{Ab}$, the morphism $f_\bullet$ is a weak equivalence if $\mathbf{U}_*(f_\bullet)$ is a weak equivalence in the Quillen model structure on $s\mathsf{Set}$. Similarly, the morphism $f_\bullet$ is a fibration in the Quillen projective model structure if $\mathbf{U}_*(f_\bullet)$ is a fibration in the Quillen model structure on $s\mathsf{Set}$. ♠

The forgetful functor is right adjoint to the free functor, so it is a right Quillen functor. Moreover, by a theorem of John Moore ([Moo55, Théorème 3] – an English version of the proof can be found in [Wei95, Lemma 8.2.8]) the forgetful functor $s\mathsf{Ab}$ to $s\mathsf{Set}$ takes any simplicial group to a fibrant simplicial set.

* Recall that geometric realisation is left adjoint to the singular functor (see [GJ09, Proposition I.2.2]). It follows that geometric realisation is a left Quillen functor, and that $\mathsf{Ho}(s\mathsf{Set}) \simeq \mathsf{Ho}(\mathsf{CW})$ for CW the category of CW-complexes.

As mentioned earlier, the normalised chain complex functor gives an equivalence between $\mathsf{Ch}^+(\mathsf{Ab})$ and $s\mathsf{Ab}$. However, even more is true. This equivalence is a Quillen equivalence if one equips $\mathsf{Ch}^+(\mathsf{Ab})$ with the projective model structure ([SS03a, Section 4]). In order to get a Quillen adjunction between $\mathsf{Ch}^+(\mathsf{Ab})$ and $s\mathsf{Set}$, one must then view the normalised chain complex functor as a left adjoint.

With the above machinery in hand, it is time to define model structures on simplicial C*-spaces and on $\mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}})$. Both model structures uses the method of Definition 1.4.12. First up is the *pointwise projective model structure* on simplicial C*-spaces, which is formed from the Quillen model structure on $s\mathsf{Set}$. By Theorem 1.4.13 and the fact that the Quillen model structure on $s\mathsf{Set}$ is proper, the pointwise projective model structure is also proper. As for the *projective model structure* on $\mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}})$, it is constructed from projective model structure on $\mathsf{Ch}^+(\mathsf{Ab})$. The latter model structure is akin to the projective model structure of $\mathsf{Ch}(\mathsf{Ab})$, and the details can be found in Definition 2.3.8.

Definition 2.3.8. Denote by I the set consisting of the chain map $0 \to \mathbf{S}^0(\mathbf{Z})$ and the inclusions $\mathbf{S}^{n-1}(\mathbf{Z}) \to \mathbf{D}^n(\mathbf{Z})$ for $n > 0$, and let J be the set of chain maps $0 \to \mathbf{D}^n(\mathbf{Z})$ for $n > 0$. There is a cofibrantly generated model structure on $\mathsf{Ch}^+(\mathsf{Ab})$, the *projective model structure*, where the fibrations are the J-injectives, the cofibrations are I-cofibrations, and weak equivalences are chain maps inducing an isomorphism on all homology groups. For more details on the projective model structure see [DS95, Section 7], where fibrations and cofibrations are also described using epi- and monomorphisms. More precisely, a chain map is a cofibration if and only if it is a levelwise monomorphism with levelwise projective cokernel, while a chain map is a fibration if and only if it is an epimorphism in all positive degrees. ♠

By Proposition 1.2.8 and Lemma 2.1.3 the vertical maps of diagram $(*)$ are isomorphisms. Thus they furnish $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Ch}^+(\mathsf{Ab})]$ with a projective model structure where a morphism $\eta \colon \mathrm{F} \to \mathrm{G}$ is a weak equivalence (resp. fibration) if η_C is a weak equivalence (resp. fibration) for all C*-algebras C. For the rightmost part of the diagram $(*)$, the projective pointwise model structure on $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}]$ defines a model structure on $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Ab}]$ by requiring that the induced free-forgetful adjunction is a Quillen adjunction.

Proposition 2.3.9. *Equip* $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}]$ *with the pointwise projective model structure and* $\mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}})$ *with the projective model structure. There is a Quillen adjunction*

$$(\mathbf{N} \circ \mathbf{F}_*)_* \colon [\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}] \rightleftarrows \mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}}, \mathsf{Ab}]_{\mathsf{Ab}}) \colon (\mathbf{U}_* \circ \mathbf{L})_*,$$

and a similar Quillen adjunction in the pointed setting:

$$(\mathbf{N} \circ \mathbf{F}_*)_* \colon [\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*] \rightleftarrows \mathsf{Ch}^+([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}) \colon (\mathbf{U}_* \circ \mathbf{L})_*.$$

Proof. The result follows by viewing the normalised chain complex functor as a left Quillen functor. Since $\mathbf{F}_*$ is a left Quillen functor it follows that the composition $\mathbf{N} \circ \mathbf{F}_*$ is a left Quillen functor, and so is $(\mathbf{N} \circ \mathbf{F}_*)_*$ by Lemma 1.4.23. □

Note that the restriction of the projective model structure on $\mathsf{Ch}(\mathsf{Ab})$ to the connective chain completenesses does not coincide with the model structure of Definition 2.3.8. However, the inclusion of $\mathsf{Ch}^+(\mathsf{Ab})$ into $\mathsf{Ch}(\mathsf{Ab})$ does preserve weak equivalences and cofibrations as Lemma 2.3.11 shows.

Definition 2.3.10. The *truncation functor* $\tau_0 \colon \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{Ch}^+(\mathsf{Ab})$ is given by $\tau_0(C)_n = C_n$ for $n > 0$, $\tau_0(C)_0 = \ker \partial_0$ for $n = 0$ and $\tau_0(C)_n = 0$ for $n < 0$. The effect of τ_0 on morphisms and boundary maps are the obvious ones. ♠

Lemma 2.3.11. *Let $\iota \colon \mathsf{Ch}^+(\mathsf{Ab}) \to \mathsf{Ch}(\mathsf{Ab})$ be the inclusion. There is a Quillen adjunction*

$$\iota \colon \mathsf{Ch}^+(\mathsf{Ab}) \rightleftarrows \mathsf{Ch}(\mathsf{Ab}) \colon \tau_0.$$

Proof. Clearly, if A and B are chain complexes with A connective, then there is a natural isomorphism $\mathrm{Hom}_{\mathsf{Ch}^+(\mathsf{Ab})}(A, \iota(B)) \simeq \mathrm{Hom}_{\mathsf{Ch}(\mathsf{Ab})}(A, \tau_0(B))$. It is equally clear that the inclusion $\mathsf{Ch}^+(\mathsf{Ab})$ into $\mathsf{Ch}(\mathsf{Ab})$ does preserve weak equivalences. Thus it only remains to show that the inclusion preserves cofibrations. However, this follows since a chain map $f \colon A \to A'$ of $\mathsf{Ch}^+(\mathsf{Ab})$ is a cofibration if it is a levelwise monomorphism with levelwise projective cokernel. It follows that $\iota(f)$ is a levelwise split monomorphism. Moreover, since A' is bounded below, the cokernel of $\iota(f)$ is cofibrant by [Hov99, Lemma 2.3.6]. Consequentially $\iota(f)$ is a cofibration by [Hov99, Proposition 2.3.9]. □

Using Lemma 2.3.11 the following corollary of Proposition 2.3.9 is immediate.

Corollary 2.3.12. *The Dold–Kan correspondence yields a Quillen adjunction*

$$(\mathbf{N} \circ \mathbf{F}_*)_* \colon [\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*] \rightleftarrows \mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}) \colon (\mathbf{U}_* \circ \mathbf{L} \circ \tau_0)_*$$

(where the inclusion functor is suppressed from the notation). □

Similarly to Section 2.2, Østvær uses left Bousfield localisation to bring the pointwise projective model structure closer to KK-theory in [Øst10]. The next results shows that the there is a Quillen adjunction between the various localised model structures on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ and $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$.

Note that in [Øst10] the theory is based on the axiomatic framework from [Cun98]. The only difference between these axioms and the ones of Proposition 2.1.7 concerns the exactness axiom. The approach of [Cun98] uses contractive completely positive split extensions (cf. the last part of Section 2.2), and as such it is based on KK-theory's universal property with respect to homotopy invariance, matrix stability and half-exactness for cpc-split extensions. Since Østvær uses slightly different axioms for KK-theory, a modification of the work in [Øst10] has to be utilised.

One way to compare the localizations of Section 2.2 with the work of Østvær is to just replace the exactness axiom used in [Øst10]. The resulting theory still works, as the required localisation now follows along the lines of [Øst10, Sections 3.3 and 3.4], and in this case the corresponding version of Proposition 2.3.13 follows along the lines of Proposition 2.3.14.

Another method is to use the approach of [Øst10, Section 3.2] with split exact sequences. The disadvantage of this method is that it does not give KK-theory. Still, there is a Quillen adjunction between that approach and the method used in Section 2.2 as Proposition 2.3.13 shows.

A final method would be to use the axiom scheme in [Cun97] in Section 2.2. However, in this case the localisation theory of Section 2.2 would be a bit trickier. Moreover, the proof of Proposition 2.3.13 would also be more complicated as the map s in the proof of Proposition 2.3.13 would no longer be a C*-algebra morphism.

Proposition 2.3.13. *The Dold–Kan correspondence gives a Quillen adjunction between the exact projective model structure on* $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ *and the split exact model structure on* $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Proof. Since $(\mathbf{N} \circ \mathbf{F}_*)_*$ preserves cofibrations it is enough to show that it also preserves acyclic cofibrations. So assume $h\colon X \to Y$ is an exact projective weak equivalence in $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ and that K is $\mathscr{T}_{CP}$-flasque. In order to show that

$$(\mathbf{N} \circ \mathbf{F}_*)_*(h)^*\colon \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(Y), \mathrm{K}\Big) \to \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(X), \mathrm{K}\Big)$$

is a bijection, first note that if $Z \in [\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ then $(\mathbf{N} \circ \mathbf{F}_*)_*(Z)$ is a connective chain complex which is free in all degrees. By Proposition 1.3.7

$$\operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(Z), \mathrm{K}\Big) \simeq \operatorname{Hom}_K\Big((\mathbf{N} \circ \mathbf{F}_*)_*(Z), \mathrm{K}\Big),$$

so without loss of generality K is also 0 in all negative degrees.

Since K is fibrant, it follows from the derived adjunction of Proposition 1.4.21 that

$$(\mathbf{N} \circ \mathbf{F}_*)_*(h)^*\colon \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(Y), \mathrm{K}\Big) \to \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(X), \mathrm{K}\Big)$$

is a bijection if and only if

$$\mathrm{Q}(h)^*\colon \operatorname{Hom}_{\mathsf{Ho}}\Big(\mathrm{Q}(Y), (\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\Big) \to \operatorname{Hom}_{\mathsf{Ho}}\Big(\mathrm{Q}(X), (\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\Big)$$

is a bijection (here Ho refers to the homotopy category of $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ in the projective model category structure, Q is the cofibrant replacement functor, and $(\mathbf{U}_* \circ \mathbf{L} \circ \tau_0)_*(\mathrm{K}) \simeq (\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ since K is a connective chain complex).

Recall that if Z is a projective fibrant object of $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ and

$$\mathrm{Q}(h)^*\colon \operatorname{Hom}_{[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]}\big(\mathrm{Q}(Y), Z\big) \to \operatorname{Hom}_{[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]}\big(\mathrm{Q}(X), Z\big)$$

is a projective weak equivalence, then the induced morphism $\mathrm{Q}(h)^*\colon \operatorname{Hom}_{\mathsf{Ho}}(\mathrm{Q}(Y), Z) \to \operatorname{Hom}_{\mathsf{Ho}}(\mathrm{Q}(X), Z)$ is an isomorphism [Hir03, Proposition 9.5.10 (2) and Proposition 9.5.24 (2)]. By the definition of a projective exact weak equivalence, [Øst10, Definition 3.30], the result follows if $(\mathbf{U}_*\circ\mathbf{L})_*(\mathrm{K})$ is flasque in the sense of [Øst10, Section 3.2]. So assume $A' \xrightarrow{f} A \underset{s}{\overset{g}{\rightleftarrows}} A''$ is a split extension of C*-algebras and consider the diagram

$$\begin{tikzcd}
\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A') \arrow[r, "\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(f)"] \arrow[d] & \mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A) \arrow[d, "\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(g)"] \\
* \arrow[r] & \mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A'').
\end{tikzcd}$$

The aim is to show that the above diagram is homotopy Cartesian. Since

$$\mathrm{K}(g)\circ\mathrm{K}(s) = \mathrm{K}(g\circ s) = \mathrm{id}_{\mathrm{K}(A'')}$$

the morphism $\mathrm{K}(g)$ is a levelwise surjection. Thus $\mathrm{K}(g)$ is a fibration in the projective model structure on $\mathsf{Ch}^+(\mathsf{Ab})$, and by the Dold–Kan correspondence $\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(g)$ is a fibration in the Quillen model structure of $s\mathsf{Set}_*$. By [GJ09, Lemma II.8.16] it is therefore enough to show that the induced morphism

$$\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A') \to * \times_{\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A'')} \mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A)$$

is a weak equivalence. Observe that $\mathbf{U}_*\circ\mathbf{L}$ is a right adjoint, so it preserves all small limits. Using the fact $\mathbf{U}_*\circ\mathbf{L}(0) = *$, this implies that

$$* \times_{\mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A'')} \mathbf{U}_*\circ\mathbf{L}\circ\mathrm{K}(A) \simeq \mathbf{U}_*\circ\mathbf{L}\big(0\times_{\mathrm{K}(A'')}\mathrm{K}(A)\big) \simeq \mathbf{U}_*\circ\mathbf{L}\big(\ker\mathrm{K}(g)\big).$$

Since the Quillen projective model structure on $s\mathsf{Ab}$ is rigged such that $\mathbf{U}_*$ takes weak equivalences to weak equivalences, it is enough to show that the induced morphism

$$\mathbf{L}\circ\mathrm{K}(A') \to \mathbf{L}\big(\ker\mathrm{K}(g)\big)$$

is a weak equivalence. Lastly, the Dold–Kan correspondence gives an equivalence of categories $\mathsf{Ch}^+(\mathsf{Ab}) \to s\mathsf{Ab}$, which is also a Quillen equivalence. Thus the functor $\mathbf{L}$ preserves weak equivalences, so it is enough to show that $\mathrm{K}(A') \to \ker\mathrm{K}(g)$ is a weak equivalence in $\mathsf{Ch}^+(\mathsf{Ab})$, i.e. it is a quasi-isomorphism.

Observe that by viewing $\ker\mathrm{K}(g)$ as a subgroup of $\mathrm{K}(A)$, the induced morphism $\mathrm{K}(A') \to \ker\mathrm{K}(g)$ is given by $\mathrm{K}(f)$ (with its codomain restricted to $\ker\mathrm{K}(g)$). Another observation is that since $\mathrm{K} = (\mathrm{K}_n, \partial_n)$ is $\mathscr{T}_{CP}$-flasque, the morphism

$$([f]^{\mathrm{Z}}\times[s]^{\mathrm{Z}})^*\colon\ \mathrm{Hom}_D([A']^{\mathrm{Z}}\oplus[A'']^{\mathrm{Z}}, \Sigma^i\mathrm{K}) \to \mathrm{Hom}_D([A]^{\mathrm{Z}}, \Sigma^i\mathrm{K})$$

is an isomorphism for all i. Moving the direct sum out, and using Proposition 1.3.7 this implies that the morphism

$$[f]^{\mathrm{Z}*} + [s]^{\mathrm{Z}*}\colon\ \mathrm{Hom}_K([A']^{\mathrm{Z}}, \Sigma^i\mathrm{K}) \oplus \mathrm{Hom}_K([A'']^{\mathrm{Z}}, \Sigma^i\mathrm{K}) \to \mathrm{Hom}_K([A]^{\mathrm{Z}}, \Sigma^i\mathrm{K}) \qquad (*)$$

is an isomorphism for all i.

The morphism $\mathrm{K}(f)$ induces a monomorphism in homology: Assume $x \in \mathrm{K}_n(A')$ is such that $(\partial_n)_{A'}(x) = 0$ and $\mathrm{K}_n f(x) = (\partial_{n+1})_{A'}(y)$ for some $y \in \mathrm{K}_{n+1}(A)$. Using the

Yoneda lemma x corresponds to a natural transformation $\theta\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A',\_) \to \mathrm{K}_n$ with $\partial_n \circ \theta = 0$ and $\theta \circ f^* = \partial_{n+1} \circ \zeta$ for some $\zeta\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A,\_) \to \mathrm{K}_{n+1}$. However, this is the same as saying that x corresponds to a chain map $\theta'\colon [A']^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}$ such that $[f]^{\mathrm{Z}^*}(\theta')\colon [A]^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}$ is null-homotopic. Since $[f]^{\mathrm{Z}^*}$ is a monomorphism θ' must be null-homotopic, i.e. $x = \theta_{A'}(\mathrm{id}_{A'})$ is in the image of $(\partial_{n+1})_{A'}$.
The morphism $\mathrm{K}(f)$ *induces an epimorphism in homology:* Assume $z \in \ker \mathrm{K}_n(g)$ is such that $(\partial_n)_A(z) = 0$. By the Yoneda lemma z corresponds to a natural transformation $\eta\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A,\_) \to \mathrm{K}_n$ such that both $\partial_n \circ \eta\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A,\_) \to \mathrm{K}_{n-1}$ and $\eta \circ g^*\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A'',\_) \to \mathrm{K}_n$ factors through 0. Thus η corresponds to a morphism $\eta'\colon [A]^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}$ such that $[g]^{\mathrm{Z}^*}(\eta') = 0$. By the isomorphism in $(*)$ there are

$$\chi'\colon [A']^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}, \quad \xi'\colon [A'']^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}, \quad \beta\colon \operatorname{Hom}_{\mathsf{C^*\text{-}alg}_{\mathsf{Ab}0}}(A,\_) \to \mathrm{K}_{n+1}$$

such that $\eta = \eta'_n = \partial_{n+1} \circ \beta + [f]^{\mathrm{Z}^*}(\chi')_n + [s]^{\mathrm{Z}^*}(\xi')_n$. Moreover,

$$\begin{aligned} 0 = [g]^{\mathrm{Z}^*}(\eta')_n &= \partial_{n+1} \circ \beta \circ g^* + [g]^{\mathrm{Z}^*} \circ [f]^{\mathrm{Z}^*}(\chi')_n + [g]^{\mathrm{Z}^*} \circ [s]^{\mathrm{Z}^*}(\xi')_n \\ &= \partial_{n+1} \circ \beta \circ g^* + [g \circ s]^{\mathrm{Z}^*}(\xi')_n = \partial_{n+1} \circ \beta \circ g^* + \xi'_n \end{aligned}$$

whence $[s]^{\mathrm{Z}^*}(\xi')_n = -\partial_{n+1} \circ \beta \circ g^* \circ s^*$ and $\eta = \chi'_n \circ f^* + \partial_{n+1} \circ \beta - \partial_{n+1} \circ \beta \circ (s \circ g)^*$. Let $w = (\chi'_n)_{A'}(\mathrm{id}_{A'}) \in \mathrm{K}_n(A')$ and note that $(\partial_n)_{A'}(w) = 0$ since $\chi'\colon [A']^{\mathrm{Z}} \to \Sigma^{-n}\mathrm{K}$. The result now follows by the fact that $z - \mathrm{K}(f)_n(w)$ lies in $\operatorname{im}(\partial_{n+1})_A$. ☐

Proposition 2.3.14. *The Dold–Kan correspondence gives Quillen adjunctions between*

1. *the matrix invariant model structure on* $[\mathsf{C^*\text{-}alg}, s\mathsf{Set}_*]$ *and the split exact and matrix stable model structure on* $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$, *and*
2. *the homotopy invariant model structure on* $[\mathsf{C^*\text{-}alg}, s\mathsf{Set}_*]$ *and the split exact, matrix stable and homotopy invariant model structure on* $\mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Proof. In both cases $(\mathbf{N} \circ \mathbf{F}_*)_*$ preserves cofibrations so it is enough to show that it also preserves acyclic cofibrations. Observe that due to the use of left Bousfield localisation, the cofibrant replacement functor Q on $\mathsf{C^*\text{-}spc}_*$ with the pointwise projective model structure coincides with the cofibrant replacement functors of $\mathsf{C^*\text{-}spc}_*$ in the exact projective, matrix invariant and homotopy invariant model structures.
The matrix stable case: Let $f\colon X \to Y$ be a matrix stable projective weak equivalence in $[\mathsf{C^*\text{-}alg}, s\mathsf{Set}_*]$ and K a $\mathscr{T}_{CP} \cup \mathscr{T}_S$-flasque object. The aim is to show that

$$(\mathbf{N} \circ \mathbf{F}_*)_*(f)^*\colon \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(Y), \mathrm{K}\Big) \to \operatorname{Hom}_D\Big((\mathbf{N} \circ \mathbf{F}_*)_*(X), \mathrm{K}\Big)$$

is a bijection. Using the same strategy as the proof of Proposition 2.3.13 it is enough to show that

$$\mathrm{Q}(f)^*\colon \operatorname{Hom}_{\mathsf{Ho}}\Big(\mathrm{Q}(Y), (\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\Big) \to \operatorname{Hom}_{\mathsf{Ho}}\Big(\mathrm{Q}(X), (\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\Big)$$

is a bijection. Thus, by the definition of a matrix invariant weak equivalence it is enough to show that $(\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ is matrix exact projective fibrant, i.e. $(\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ is exact projective fibrant and for all separable C*-algebras A the morphism

$$\big((\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\big)(k_A)\colon \big((\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\big)(A) \to \big((\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\big)(A \otimes_\sigma \mathbb{K})$$

is a weak equivalence of simplicial sets. Since K is $\mathscr{T}_{CP}$-flasque, K is fibrant whence $(\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ is also fibrant. By the proof of Proposition 2.3.13, the $\mathscr{T}_{CP}$-flasqueness of K implies that $(\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ is flasque. Thus $(\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})$ is exact projective fibrant, so in order to show that $\big((\mathbf{U}_* \circ \mathbf{L})_*(\mathrm{K})\big)(k_A)$ is a weak equivalence of simplicial sets it is enough to show that $\mathrm{K}(k_A)\colon \mathrm{K}(A) \to \mathrm{K}(A \otimes_\sigma \mathbb{K})$ is a quasi-isomorphism.

Applying $\mathrm{Hom}_D(\_, \Sigma^n \mathrm{K})$ to $[A \otimes_\sigma \mathbb{K}]^{\mathrm{Z}} \to [A]^{\mathrm{Z}} \to \mathrm{cone}\,([k_A]^{\mathrm{Z}}) \to \Sigma[A]^{\mathrm{Z}}$ yields an isomorphism $\mathrm{Hom}_D([A]^{\mathrm{Z}}, \Sigma^n \mathrm{K}) \to \mathrm{Hom}_D([A \otimes_\sigma \mathbb{K}]^{\mathrm{Z}}, \Sigma^n \mathrm{K})$ by the $\mathscr{T}_S$-flasqueness of $\mathrm{K} = (\mathrm{K}_n, \partial_n)$. Since Proposition 1.3.7 is applicable, there is an isomorphism

$$\mathrm{Hom}_K(k_A, \Sigma^n \mathrm{K})\colon \mathrm{Hom}_K([A]^{\mathrm{Z}}, \Sigma^n \mathrm{K}) \to \mathrm{Hom}_K([A \otimes_\sigma \mathbb{K}]^{\mathrm{Z}}, \Sigma^n \mathrm{K}),$$

so by the Yoneda lemma $\mathrm{K}_n(k_A)$ induces an isomorphism

$$\ker(\partial_n)_A / \operatorname{im}(\partial_{n+1})_A \to \ker(\partial_n)_{A \otimes_\sigma \mathbb{K}} / \operatorname{im}(\partial_{n+1})_{A \otimes_\sigma \mathbb{K}}.$$

It follows that $\mathrm{K}(k_A)$ is a quasi-isomorphism.
The homotopy invariant case: The proof is basically the same as the matrix invariant case. □

2.4 Spectra of C*-spaces

The category $\mathsf{Ch}^+(\mathsf{Ab})$ has a slight deficiency since the suspension functor Σ lacks an inverse. This is easily remedied by looking at the category $\mathsf{Ch}(\mathsf{Ab})$ instead. Correspondingly, $s\mathsf{Set}_*$ also has a suspension functor (given by $\_ \wedge S^1_\bullet$) which lacks an inverse. In order to remedy the simplicial situation one must look at spectra of simplicial sets.

Definition 2.4.1. The *simplicial circle* or *simplicial* 1*-sphere* is the simplicial set $S^1_\bullet = \Delta^1_\bullet / \partial \Delta^1_\bullet$. Note that the unique 0-simplex of $S^1_\bullet$ make the simplicial 1-sphere into pointed simplicial sets. For $n > 1$ the *simplicial* n*-sphere* is the simplicial set

$$S^n_\bullet = \underbrace{S^1_\bullet \wedge \cdots \wedge S^1_\bullet}_{n \text{ times}}.$$

A *simplicial spectrum* is a collection $(X_{n,\bullet}, \sigma_{n,\bullet})_{n \geq 0}$ of pointed simplicial sets $X_{n,\bullet}$ and pointed simplicial maps (called structure maps) $\sigma_{n,\bullet}\colon S^1_\bullet \wedge X_{n,\bullet} \to X_{n+1,\bullet}$. Given two simplicial spectra $(X_{n,\bullet}, \sigma_{n,\bullet})$ and $(X'_{n,\bullet}, \sigma'_{n,\bullet})$, a *morphism of simplicial spectra* $(f_{n,\bullet})\colon (X_{n,\bullet}, \sigma_{n,\bullet}) \to (X'_{n,\bullet}, \sigma'_{n,\bullet})$ is a collection $(f_{n,\bullet})_{n \geq 0}$ of morphisms $f_{n,\bullet}\colon X_{n,\bullet} \to X'_{n,\bullet}$ in $s\mathsf{Set}_*$ such that the diagram

$$\begin{tikzcd}
S^1_\bullet \wedge X_{n,\bullet} \arrow[r, "\sigma_{n,\bullet}"] \arrow[d, "\mathrm{id}_{S^1_\bullet} \wedge f_{n,\bullet}"'] & X_{n+1,\bullet} \arrow[d, "f_{n+1,\bullet}"] \\
S^1_\bullet \wedge X'_{n,\bullet} \arrow[r, "\sigma'_{n,\bullet}"] & X'_{n+1,\bullet}
\end{tikzcd}$$

commutes for all $n \geq 0$. The category of simplicial spectra is denoted by Spt.

Given a pointed simplicial set $X_\bullet$, the *suspension spectrum* $\Sigma^\infty X_\bullet$ has $(\Sigma^\infty X_\bullet)_n = S^n_\bullet \wedge X_\bullet$ and $\sigma_{n,\bullet} = \mathrm{id}_{S^{n+1}_\bullet \wedge X_\bullet}$. Similarly, given a morphism of simplicial sets $f_\bullet \colon X_\bullet \to Y_\bullet$, there is a morphism $\Sigma^\infty f_\bullet \colon \Sigma^\infty X_\bullet \to \Sigma^\infty Y_\bullet$ where $(\Sigma^\infty f_\bullet)_n = \mathrm{id}_{S^n_\bullet} \wedge f_\bullet$. This construction gives a functor $\Sigma^\infty \colon s\mathsf{Set}_* \to \mathsf{Spt}$.

Let $X = (X_{n,\bullet}, \sigma_{n,\bullet})_{n \geq 0}$ be a simplicial spectrum. The functor $\mathbf{F} \colon \mathsf{Spt} \to \mathsf{Spt}$ has $\mathbf{F}(X)_n = \mathbf{F}_*(X_{n,\bullet})$. If $e_x \in \mathbf{F}_*(X_{n,\bullet})$ denotes the generator corresponding to $x \in X_{n,\bullet}$ then the structure map $\sigma'_{n,\bullet} \colon S^1_\bullet \wedge \mathbf{F}(X)_n \to \mathbf{F}(X)_{n+1}$ is given by $l \wedge e_x \mapsto e_{\sigma_{n,\bullet}(l \wedge x)}$. ♠

In the setting of spectra, an analogue of a simplicial abelian group is a naive module over the Eilenberg–Mac Lane spectrum of the integers. For instance, $\mathbf{F}(X)$ will be a naive module over the Eilenberg–Mac Lane spectrum of $\mathbf{Z}$ for any simplicial spectrum X.

Definition 2.4.2. A *symmetric spectrum* is a simplicial spectrum $(X_{n,\bullet}, \sigma_{n,\bullet})_{n \geq 0}$ such that

- there is a basepoint preserving left action of the symmetric group Σ_n on $X_{n,\bullet}$, and
- the morphisms $\sigma_{p+q-1,\bullet} \circ \cdots \circ \sigma_{q,\bullet} \colon S^p_\bullet \wedge X_{q,\bullet} \to X_{p+q,\bullet}$ are $\Sigma_p \times \Sigma_q$-equivariant (where Σ_p acts on $S^p_\bullet$ by permuting the coordinates).

The symmetric spectra together with *morphisms of symmetric spectra*, that is morphisms of simplicial spectra $(f_{n,\bullet}) \colon (X_{n,\bullet}, \sigma_{n,\bullet}) \to (X'_{n,\bullet}, \sigma'_{n,\bullet})$ such that $f_{n,\bullet}$ is Σ_n equivariant, form the category Spt^Σ.

The *Eilenberg–Mac Lane spectrum of* $\mathbf{Z}$, $\mathrm{H}\mathbf{Z}$, is the symmetric spectrum with mth space $\mathbf{F}_*(S^m_\bullet)$, structure maps of the form $l \wedge e_x \mapsto e_{l \wedge x}$, and where Σ_m acts by permuting the simplicial circles.

A *naive module over* $\mathrm{H}\mathbf{Z}$ is a spectrum $(X_{n,\bullet}, \sigma_{n,\bullet})$ where each $X_{n,\bullet}$ is a simplicial abelian group and the structure maps can be extended to simplicial group homomorphisms* $(\mathrm{H}\mathbf{Z})_m \wedge X_{n,\bullet} \to X_{m+n,\bullet}$. Denote by nvmod-H$\mathbf{Z}$ the category of naive modules over H$\mathbf{Z}$. ♠

Recall that spectra should solve the problem of inverting the suspension of simplicial sets. The spectrum corresponding to the simplicial set $X_\bullet$ is the suspension spectrum $\Sigma^\infty X_\bullet$. A natural candidate for a delooping of $\Sigma^\infty X_\bullet$ is the spectrum $X'_\bullet$ with $X'_{n,\bullet} = X_{n-1,\bullet}$ for $n > 0$ and $X'_{0,\bullet} = *$. However, if one suspends $X'_\bullet$ (i.e. apply the suspension functor at each degree) then the result differs from $X_\bullet$. By using a twist, there is a morphism from the suspension on $X'_\bullet$ to $X_\bullet$, but this needs not be an isomorphism. The stable model structure solves this problem.

Definition 2.4.3. Let $f \colon X \to Y$ be a morphism of simplicial spectra. Call the morphism f a *stable weak equivalence* if it induces monomorphisms stable homotopy groups (i.e. the induced maps $\operatorname{colim}_{n \to \infty} \pi_{n+k}(X_{n,\bullet}) \to \operatorname{colim}_{n \to \infty} \pi_{n+k}(Y_{n,\bullet})$ are isomorphisms

* Note that for simplicial abelian groups $X_\bullet$, $Y_\bullet$ their monoidal product is given by $(X_\bullet \wedge Y_\bullet)_n = X_n \otimes Y_n$.

for all $k \geq 0$). The morphism f is a *stable fibration* if it is a levelwise fibration and for any $n \geq 0$ the diagram

$$\begin{tikzcd} X_{n,\bullet} \arrow[r, "\eta_{X_n}"] \arrow[d, "f_n"] & (\Omega^\infty \operatorname{Ex}^\infty X)_{n,\bullet} \arrow[d, "\Omega^\infty \operatorname{Ex}^\infty f_n"] \\ Y_{n,\bullet} \arrow[r, "\eta_{Y_n}"] & (\Omega^\infty \operatorname{Ex}^\infty Y)_{n,\bullet} \end{tikzcd}$$

is a homotopy fiber square (see [BF78, p. 85] and [GJ09, p. 500]). In the *stable model structure* on Spt the weak equivalences are stable weak equivalences, while the fibrations are stable fibrations.

A morphism $f\colon X \to Y$ of naive modules over $\mathrm{H}\mathbf{Z}$ is a *stable weak equivalence* (resp. *stable fibration*) if it is a stable weak equivalence (resp. fibration) when viewed as a morphism in Spt. In the *stable model structure* on nvmod-$\mathrm{H}\mathbf{Z}$ the weak equivalences are stable weak equivalences, while the fibrations are stable fibrations.

There is another candidate for a model structure on Spt. In the *levelwise model structure* a morphism $f\colon X \to Y$ is a weak equivalence (resp. fibration) if the morphism $f_{n,\bullet}\colon X_{n,\bullet} \to Y_{n,\bullet}$ is a weak equivalence (resp. fibration) in the Quillen projective model structure on $s\mathsf{Set}_*$. By [Hov01, Corollary 3.5]*, one obtains the stable model structure from the levelwise model structure by left Bousfield localisation on a certain set of morphisms.

As mentioned earlier the Eilenberg–Mac Lane spectrum of $\mathbf{Z}$ is akin to $\mathbf{Z}$. One aspect of this is the notion of the derived category of the Eilenberg–Mac Lane spectrum of $\mathbf{Z}$ (or more generally any ring spectrum). In the article [Rob87] Alan Robinson gave an equivalence between the derived category of a ring R, $D(\mathrm{mod}_R)$, to the derived category of the Eilenberg–Mac Lane spectrum of R.

This equivalence was later generalised to a zig-zag of Quillen equivalences by Stefan Schwede and Brooke Shipley in [SS03b, Appendix B]. One step in their generalisation was a stable version of the Dold–Kan correspondence, which hinges on a natural transformation $\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L}(\_) \to \mathbf{L} \circ \Sigma$. In order to get a better grasp of the simplicial circle, the following notation is helpful.

Definition 2.4.4. For a simplicial set $X_\bullet$, the set of *non-degenerate n-simplicies* is $X^\sharp_n = X_n \setminus \{s^{n-1}_i(y) \mid y \in X_{n-1}, 0 \leq i \leq n-1\}$. If $X_\bullet$ is a pointed simplicial set, the basepoint in each space $X_{n,\bullet}$ is considered a degenerate n-simplex. Moreover, in this case $X^*_n = X_n \setminus \{*\}$ where $*$ is the basepoint. ♠

Example 2.4.5. Since $\partial\Delta^1_n = \mathrm{Hom}_\Delta(\mathbf{n}, \mathbf{1}) = \{f_0, f_1\}$ where $f_0(t) = 0$, $f_1(t) = 1$ for all $t \in \mathbf{n}$, it follows that $(S^1_n)^* = \{f\colon \mathbf{n} \to \mathbf{1} \mid f(0) = 0, f(n) = 1\}$. Thus there is a bijection $l\_\colon \{1, \ldots, n\} \to (S^1_n)^*$,

$$l_k(a) = \begin{cases} 0 & a < k, \\ 1 & a \geq k, \end{cases}$$

* Recall that all simplicial sets are cofibrant, and note that the preprint "Localization of model categories" cited in [Hov01] has become the book [Hir03].

and using this it is possible to describe the face and degeneracy maps as follows:

$$d_j^n(l_k) = \begin{cases} l_k & k \le j, k \le n-1, \\ l_{k-1} & k > j, k \ne 1, \\ * & k = 1, j = 0, \\ * & k = n, j = n, \end{cases} \qquad s_i^n(l_k) = \begin{cases} l_k & k \le i, \\ l_{k+1} & k > i. \end{cases}$$

Consequentially, if $X_\bullet$ is a pointed simplicial set then $(S^1_\bullet \wedge X_\bullet)^*_n \simeq \{1, \dots, n\} \times X^*_n$ and for $0 \le j \le n$

$$d_j^n(l_k, x) = \begin{cases} \big(l_k, d_j^n(x)\big) & k \le j, k \ne n, d_j^n(x) \ne *, \\ \big(l_{k-1}, d_j^n(x)\big) & k > j, k \ne 1, d_j^n(x) \ne *, \\ * & d_j^n(x) = *, \\ * & k = 1, j = 0, \\ * & k = n, j = n, \end{cases} \qquad s_j^n(l_k, x) = \begin{cases} \big(l_k, s_j^n(x)\big) & k \le j, \\ \big(l_{k+1}, s_j^n(x)\big) & k > j. \end{cases}$$

This implies that for $n > 0$ the degenerate n simplicies of $S^1_\bullet \wedge X_\bullet$ are

$$D_n = \left\{ \big(l_k, s_i^{n-1}(x)\big) \;\middle|\; \begin{matrix} 0 < k \le i < n \\ x \in X_{n-1} \end{matrix} \right\} \cup \left\{ \big(l_{k+1}, s_i^{n-1}(x)\big) \;\middle|\; \begin{matrix} 0 \le i < k < n \\ x \in X_{n-1} \end{matrix} \right\}.$$

If $\big(l_l, s_k^{n-1}(x)\big) \notin D_n$, then $l > k$ and $k + 1 \ge l$ so $l = k + 1$. Consider the case where $x = s_j^{n-2}(y)$ i.e. look at $\big(l_{k+1}, s_k^{n-1} \circ s_j^{n-2}(y)\big)$. There are two possibilities. If $k \le j$ then

$$\big(l_{k+1}, s_k^{n-1} \circ s_j^{n-2}(y)\big) = \big(l_{k+1}, s_{j+1}^{n-1} \circ s_k^{n-2}(y)\big) \in D_n$$

while if $k > j$ then

$$\big(l_{k+1}, s_k^{n-1} \circ s_j^{n-2}(y)\big) = \big(l_{k+1}, s_j^{n-1} \circ s_{k-1}^{n-2}(y)\big) \in D_n$$

so $\big(l_{k+1}, s_k^{n-1} \circ s_j^{n-2}(y)\big) \in D_n$ in both cases. Thus

$$(S^1_\bullet \wedge X_\bullet)^\sharp_n = \{l_1, \dots, l_n\} \times X^\sharp_n \cup \left\{ \big(l_{k+1}, s_k^{n-1}(x)\big) \;\middle|\; x \in X^\sharp_{n-1}, 0 \le k \le n-1 \right\}. \qquad \clubsuit$$

Definition 2.4.6. For a simplicial abelian group $X_\bullet$ the ith *back face* is

$$\overrightarrow{d}^{\,i} \colon X_n \to X_{n-i}, \quad x \mapsto d_{n+1-i}^{n+1-i} \circ \cdots \circ d_n^n(x)$$

and the ith *front face* is

$$\overleftarrow{d}^{\,i} \colon X_n \to X_{n-i}, \quad x \mapsto d_0^{n+1-i} \circ \cdots \circ d_0^n(x).$$

The *non-normalised chain complex* of $X_\bullet$ is the chain complex $\mathbf{C}(X_\bullet) = (X_n, \partial_n)$ where $\partial_n = \sum_{i=0}^n (-1)^i d_i^n$. If $Y_\bullet$ is another simplicial abelian group, then the *Alexander–Whitney map* is the natural transformation $f^{X,Y} \colon \mathbf{C}(X_\bullet \wedge Y_\bullet) \to \mathbf{C}(X_\bullet) \otimes_T \mathbf{C}(Y_\bullet)$ defined by

$$f_n^{X,Y} \colon X_n \otimes Y_n \to \bigoplus_{i=0}^n X_i \otimes Y_{n-i}, \quad x \otimes y \mapsto \sum_{i=0}^n \overrightarrow{d}^{\,n-i}(x) \otimes \overleftarrow{d}^{\,i}(y),$$

and it descends to a natural transformation $g^{X,Y}\colon \mathbf{N}(X_\bullet \wedge Y_\bullet) \to \mathbf{N}(X_\bullet) \otimes_T \mathbf{N}(Y_\bullet)$ [Mac95, Theorem VIII.8.5 and Corollary VIII.8.6]. In the special case where $X_\bullet = \mathbf{F}_*(S^1_\bullet)$ and $Y_\bullet = \mathbf{L}(C)$ this gives a natural chain map

$$\rho_C\colon \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L}(C)\big) \to \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet)\big) \otimes_T C \simeq \Sigma C.$$

Applying the functor $\mathbf{L}$ to ρ_C yields* the natural transformation $\mathbf{L}(\rho\_)\colon \mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L}(\_) \to \mathbf{L} \circ \Sigma(\_)$. Recall the truncation functor $\tau_0\colon \mathsf{Ch}(\mathsf{Ab}) \to \mathsf{Ch}^+(\mathsf{Ab})$ and note that if $C = (C_n, \partial_n)$ then

$$\big(\Sigma \circ \tau_0 \circ \Sigma^n(C)\big)_l = \begin{cases} \big(\tau_0 \circ \Sigma^{n+1}(C)\big)_l & l > 1, \\ \ker \partial_{-n} & l = 1, \\ 0 & l < 1, \end{cases}$$

while $\big(\tau_0 \circ \Sigma^{n+1}(C)\big)_1 = C_{-n}$ and $\big(\tau_0 \circ \Sigma^{n+1}(C)\big)_0 = \ker \partial_{-(n+1)}$. Thus the natural transformation $\mathbf{L}(\rho\_)$ induces natural transformations†

$$\rho'_n\colon \mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L} \circ \tau_0 \circ \Sigma^n(\_) \to \mathbf{L} \circ \tau_0 \circ \Sigma^{n+1}(\_)$$

for all $n \geq 0$.

The *Eilenberg–Mac Lane functor* $\Phi\colon \mathsf{Ch}(\mathsf{Ab}) \to$ nvmod-H$\mathbf{Z}$ is the stable analogue of $\mathbf{L}$. It sends a chain complex C to the simplicial spectrum $(\mathbf{L} \circ \tau_0 \circ \Sigma^n(C), (\rho''_n)_C)$ (where $(\rho''_n)_C(l_k \wedge e_x) = (\rho'_n)_C(e_{l_k} \otimes e_x))$, while a chain map $f\colon C_1 \to C_2$ is sent to the morphism of simplicial spectra which at level n is $\mathbf{L} \circ \tau_0 \circ \Sigma^n(f)$. ♠

Example 2.4.7. For an abelian group A, consider the chain complex $\mathbf{S}^0(A) = A^0$. In this case $\Phi(A^0)^\sharp_{n,n} \simeq A$ while $\Phi(A^0)^\sharp_{n,k} \simeq 0$ if $n \neq k$. Thus $\Phi(A^0)_{n,\bullet} \simeq \mathbf{F}_*(\Delta^n_\bullet/\partial\Delta^n_\bullet) \wedge A^\Delta$ where A^Δ is the simplicial abelian group which is A in each degree, and where all face and degeneracy maps are identities.

The structure map $\sigma_{n,\bullet}\colon \mathbf{F}_*(S^1_\bullet) \wedge \mathbf{F}_*(\Delta^n_\bullet/\partial\Delta^n_\bullet) \wedge A^\Delta \to \mathbf{F}_*(\Delta^{n+1}_\bullet/\partial\Delta^{n+1}_\bullet) \wedge A^\Delta$ is determined by what it does in degrees n and $n+1$ (cf. Example 2.4.5). In degree n it must be the zero map since $(\Delta^{n+1}_\bullet/\partial\Delta^{n+1}_\bullet)_n = \{*\}$. In degree $n+1$ it is induced by the map $x \otimes y \mapsto \sum_{i=0}^{n+1} \overrightarrow{d}^{\,n+1-i}(x) \otimes \overleftarrow{d}^{\,i}(y) = \overrightarrow{d}^{\,n+1}(x) \otimes \overleftarrow{d}^{\,0}(y) + \overrightarrow{d}^{\,n}(x) \otimes \overleftarrow{d}^{\,1}(y)$. There is no loss of generality in assuming $x = e_{l_{k+1}}$ and $y = s^{n-1}_k(a)$ with $a \in A$ and $e_{l_{k+1}}$ the generator corresponding to l_{k+1}. Since $\overrightarrow{d}^{\,n+1}(x) \in \mathbf{F}_*(S^1_0) \simeq 0$ and $\overleftarrow{d}^{\,1}(y) = d_0 s_k(a)$ it follows that this is zero except for when $k = 1$, and that $e_{l_1} \otimes s_0(a) \mapsto e_{\mathrm{id}_{\Delta^{n+1}_\bullet}} \otimes a$. ♣

Observe that the geometric realisation of $\Delta^n_\bullet/\partial\Delta^n_\bullet$, $S^n_\bullet$ and $\partial\Delta^{n+1}_\bullet$ all are homeomorphic to the topological n-sphere. Thus they are weakly equivariant in $s\mathsf{Set}_*$, and it follows that $\mathbf{F}_*(\Delta^n_\bullet/\partial\Delta^n_\bullet) \wedge A^\Delta$, $\mathbf{F}_*(S^n_\bullet) \wedge A^\Delta$ and $\mathbf{F}_*(\partial\Delta^{n+1}_\bullet) \wedge A^\Delta$ are all weakly equivariant in $s\mathsf{Ab}$ when A is a free abelian group‡.

* Using the natural isomorphism $Z_\bullet \to \mathbf{L} \circ \mathbf{N}(Z_\bullet)$ from the Dold–Kan correspondence. † Where the functors goes from $\mathsf{Ch}(\mathsf{Ab})$ to $s\mathsf{Ab}$. ‡ Recall that $\mathbf{F}_*$ is a left Quillen functor, all objects of $s\mathsf{Set}_*$ are cofibrant and $s\mathsf{Set}$ satisfies the pushout-product axiom. Thus the result follows since $\mathbf{F}_*(X_\bullet) \wedge A^\Delta \simeq \mathbf{F}_*(X_\bullet \wedge A^\Delta_+)$ with A_+ a set of generators for A with a disjoint basepoint.

By [SS03b, Theorem B.1.11] the functor Φ is a right adjoint. Moreover [SS03b, Theorem B.1.11] says that if naive modules over $\mathrm{H}\mathbf{Z}$ are given the stable model structure (see Definition 2.4.3), this adjunction gives a Quillen equivalence.

In order for Φ to be a right adjoint, there must be a corresponding left adjoint $\Psi\colon$ nvmod-$\mathrm{H}\mathbf{Z} \to \mathsf{Ch}(\mathsf{Ab})$. The approach of [SS03b] is to show that the properties of Φ gives a left adjoint by Freyd's adjoint functor theorem. An explicit construction of Ψ, which resembles the construction of the functor D in [Shi07, p. 374], is given in Definition 2.4.10.

At the heart of the functor Ψ is the normalised chain complex functor $\mathbf{N}$, so it is necessary to know how $\mathbf{N}$ behaves with respect to suspensions. An immediate consequence of Example 2.4.5 is Lemma 2.4.8, while a straightforward computation using the simplicial identities yields Lemma 2.4.9.

Lemma 2.4.8. *Let $X_\bullet$ be a simplicial abelian group with face maps d_i^n and denote the face maps of $\mathbf{F}_*(S^1_\bullet) \wedge X_\bullet$ by $\tilde{d_i^n}$. The kernels of $\tilde{d_i^n}$ are generated by the following sets:*

$$\begin{aligned}
\ker \tilde{d_0^n} &: \left\{ e_{l_i} \otimes x \;\middle|\; i > 0, x \in \ker d_0^n \right\} \cup \left\{ e_{l_1} \otimes x \;\middle|\; x \in X_n \right\}, \\
\ker \tilde{d_n^n} &: \left\{ e_{l_i} \otimes x \;\middle|\; i < n, x \in \ker d_n^n \right\} \cup \left\{ e_{l_n} \otimes x \;\middle|\; x \in X_n \right\}, \text{ and} \\
\ker \tilde{d_k^n} &: \left\{ e_{l_i} \otimes x \;\middle|\; i \neq k, x \in \ker d_k^n \right\} \cup \left\{ (e_{l_{k+1}} - e_{l_k}) \otimes x \;\middle|\; x \in X_n \right\}
\end{aligned}$$

where $0 < k < n$ and $e_{l_{k+1}}$ is the generator corresponding to l_{k+1} of Example 2.4.5. $\square$

Lemma 2.4.9. *Let $X_\bullet$ be a simplicial abelian group, and define $\kappa_{n,k}\colon X_{n-1} \to X_n$ by $\kappa_{n,k}(x) = \sum_{i=k}^{n-1} (-1)^i s_i^{n-1}(x)$. If $x \in \bigcap_{i=0}^{n-2} \ker d_i^{n-1} = \mathbf{N}(X_\bullet)_{n-1}$ then $\kappa_{n,k}(x) \in \ker d_i^n$ for $i \neq k, n$ and $d_k^n \circ \kappa_{n,k}(x) = (-1)^k x$.* $\square$

Definition 2.4.10. The natural transformation $\kappa'\colon \Sigma\mathbf{N}(\_) \to \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \_\big)$ is given by

$$\begin{aligned}
(\kappa'_{X_\bullet})_n\colon \big(\Sigma\mathbf{N}(X_\bullet)\big)_n &\to \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge X_\bullet\big)_n \\
x &\mapsto e_{l_1} \otimes \kappa_{n,0}(x) + \sum_{k=1}^{n-1} (e_{l_{k+1}} - e_{l_k}) \otimes \kappa_{n,k}(x) \\
&= \sum_{k=0}^{n-1} (-1)^k e_{l_{k+1}} \otimes s_k^{n-1}(x).
\end{aligned}$$

Let $(X_{n,\bullet}, \sigma_{n,\bullet})$ be a naive module over $\mathrm{H}\mathbf{Z}$, and consider the diagram

$$\mathbf{N}(X_{0,\bullet}) \xrightarrow{\Sigma^{-1}\kappa'_{X_{0,\bullet}}} \Sigma^{-1}\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge X_{0,\bullet}\big) \xrightarrow{\Sigma^{-1}\mathbf{N}(\sigma_{0,\bullet})} \Sigma^{-1}\mathbf{N}(X_{1,\bullet}) \xrightarrow{\Sigma^{-2}\kappa'_{X_{1,\bullet}}} \cdots \qquad (*)$$

The functor $\Psi\colon$ nvmod-$\mathrm{H}\mathbf{Z} \to \mathsf{Ch}(\mathsf{Ab})$ sends $(X_{n,\bullet}, \sigma_{n,\bullet})$ to the colimit of the diagram $(*)$, i.e.

$$\Psi(X_{n,\bullet}, \sigma_{n,\bullet}) = \operatorname*{colim}_n \Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \qquad \spadesuit$$

Example 2.4.11. Let A be an abelian group and consider $\Sigma^{\infty}_{\min} A^{\Delta}$, the naive module over H**Z** which in degree n is $\mathbf{F}_*(\Delta^n_\bullet/\partial\Delta^n_\bullet) \wedge A^{\Delta}$ and where the structure maps are the ones described in Example 2.4.7. In this case $\mathbf{N}\big((\Sigma^{\infty}_{\min} A^{\Delta})_{n,\bullet}\big) \simeq \Sigma^n A^0$ while

$$(\kappa'_{\Sigma^{\infty}_{\min} A^{\Delta}_{n,\bullet}})_{n+1} \colon \Sigma\mathbf{N}\big((\Sigma^{\infty}_{\min} A^{\Delta})_{n,\bullet}\big)_{n+1} \to \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \big((\Sigma^{\infty}_{\min} A^{\Delta})_{n,\bullet}\big)\big)_{n+1}$$

is the map $a \mapsto \sum_{k=0}^{n} (-1)^k e_{l_{k+1}} \otimes s^n_k(a)$. It follows that $\mathbf{N}(\sigma_{n,\bullet}) \circ (\kappa'_{\Sigma^{\infty}_{\min} A^{\Delta}_{n,\bullet}})$ is the identity on A in degree $n+1$ and 0 otherwise, so $\Psi(\Sigma^{\infty}_{\min} A^{\Delta}) \simeq A^0$.

Note that $(\Sigma^{\infty}_{\min} A^{\Delta})_n = \mathbf{F}_*(\Delta^n_\bullet/\partial\Delta^n_\bullet) \wedge A^{\Delta}$ is weakly equivariant to $\mathbf{F}_*(S^n_\bullet) \wedge A^{\Delta}$, i.e. the spectra $(\Sigma^{\infty}_{\min} A^{\Delta})$ and $\Sigma^{\infty} A^{\Delta}$ are level equivariant. Since Ψ is a left Quillen functor (see Theorem 2.4.13) $\Psi(\Sigma^{\infty} A^{\Delta})$ is quasi-isomorphic to A^0. ♣

The adjunction between Φ and Ψ now follows from Lemma 2.4.12 and the ordinary Dold–Kan correspondence.

Lemma 2.4.12. *If $C = (C_n, \partial_n)$ is a chain complex, then the composite*

$$\begin{tikzcd}
\Sigma\tau_0(C) \arrow[d, equal] & & & & \tau_0\Sigma(C) \arrow[d, equal] \\
\Sigma \circ \mathbf{N} \circ \mathbf{L} \circ \tau_0(C) \arrow[rr, "\kappa'_{\mathbf{L}\tau_0 C}"] & & \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L} \circ \tau_0(C)\big) \arrow[rr, "\mathbf{N}((\rho'_0)_C)"] & & \mathbf{N} \circ \mathbf{L} \circ \tau_0 \circ \Sigma(C)
\end{tikzcd}$$

is an isomorphism in all degrees above 1 *and the inclusion of* $\ker\partial_0$ *in degree* 1.

Proof. Assume $n > 0$ and $x \in \Sigma\tau_0(C)_n$. In this case

$$\begin{aligned}
\mathbf{N}((\rho'_0)_C)_n \circ (\kappa'_{\mathbf{L}\tau_0 C})_n(x) &= (\rho_C)_n\Big(\sum_{k=0}^{n-1}(-1)^k e_{l_{k+1}} \otimes s^{n-1}_k(x)\Big) \\
&= \sum_{i=0}^{n}\sum_{k=0}^{n-1}(-1)^k \overrightarrow{d}^{\,n-i}(e_{l_{k+1}}) \otimes \overleftarrow{d}^{\,i}\big(s^{n-1}_k(x)\big) \\
&= \sum_{k=0}^{n-1}(-1)^k \overrightarrow{d}^{\,n-1}(e_{l_{k+1}}) \otimes d^1_0 \circ \big(s^{n-1}_k(x)\big)
\end{aligned}$$

since $\overrightarrow{d}^{\,n-i}(e_{l_{k+1}}) \in \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet)\big)_i$ and $\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet)\big)_i = 0$ for $i \neq 1$. Moreover, $\overrightarrow{d}^{\,n-1} = d^2_2 \circ \cdots \circ d^n_n(x)$ whence $\overrightarrow{d}^{\,n-1}(e_{l_{k+1}}) = 0$ for $k > 0$. Thus

$$\mathbf{N}((\rho'_0)_C)_n \circ (\kappa'_{\mathbf{L}\tau_0 C})_n(x) = e_{l_1} \otimes d^n_0 s^{n-1}_0(x) = e_{l_1} \otimes x \in \mathbf{N}\big(\mathbf{F}_*(S^1_\bullet)\big)_1 \otimes C_{n-1}. \qquad \square$$

Theorem 2.4.13. *The functor Ψ is left adjoint to the Eilenberg–Mac Lane functor Φ.*

Proof. Assume $X = (X_{n,\bullet}, \sigma_{n,\bullet})$ is a naive module over H**Z** and (C_n, ∂_n) is a chain complex. A morphism of naive H**Z**-modules $f \colon X \to \Phi(C)$ consists of morphisms

$(f_{n,\bullet})\colon X_{n,\bullet} \to \Phi(C)_n$ compatible with the module structure. Thus there are commutative diagrams

$$\begin{tikzcd}
\mathbf{F}_*(S^1_\bullet) \wedge X_{n,\bullet} \arrow[r, "\sigma_{n,\bullet}"] \arrow[d, "\mathrm{id}_{\mathbf{F}_*(S^1_\bullet)} \wedge f_{n,\bullet}"] & X_{n+1,\bullet} \arrow[d, "f_{n+1,\bullet}"] \\
\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L} \circ \tau_0 \circ \Sigma^n C \arrow[r, "(\rho'_n)_C"] & \mathbf{L} \circ \tau_0 \circ \Sigma^{n+1} C,
\end{tikzcd}$$

whence the diagrams

$$\begin{tikzcd}
\Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \arrow[r, "\Sigma^{-(n+1)}\kappa'_{X_{n,\bullet}}"] \arrow[d, "\Sigma^{-n}\mathbf{N}(f_{n,\bullet})"'] & \Sigma^{-(n+1)}\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge X_{n,\bullet}\big) \arrow[r, "\Sigma^{-(n+1)}\mathbf{N}(\sigma_n)"] \arrow[d, "\Sigma^{-(n+1)}\mathbf{N}(\mathrm{id}_{\mathbf{F}_*(S^1_\bullet)} \wedge f_{n,\bullet})"'] & \Sigma^{-(n+1)}\mathbf{N}(X_{n+1,\bullet}) \arrow[d, "\Sigma^{-(n+1)}\mathbf{N}(f_{n+1,\bullet})"'] \\
\Sigma^{-n}\tau_0\Sigma^n C \arrow[r, "\Sigma^{-(n+1)}\kappa'_{\mathbf{L}\tau_0\Sigma^n C}"] & \Sigma^{-(n+1)}\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L}\tau_0\Sigma^n C\big) \arrow[r, "\Sigma^{-(n+1)}\mathbf{N}((\rho'_n)_C)"] & \Sigma^{-(n+1)}\tau_0\Sigma^{n+1} C
\end{tikzcd}$$

commute. The splicing together of top rows give a sequence whose colimit is $\Psi(X)$, and the similar splicing of the bottom rows give a sequence with colimit C by Lemma 2.4.12. Thus there is an induced chain map $F\colon \Psi(X) \to C$.

If $f'\colon X \to \Phi(C)$ is another morphism of naive modules over H$\mathbf{Z}$ and $F' = F$, then the chain maps $\mathbf{N}(f_{n,\bullet})$ and $\mathbf{N}(f'_{n,\bullet})$ have to agree by Lemma 2.4.12. The Dold–Kan correspondence then implies $f_{n,\bullet} = f'_{n,\bullet}$ for all n whence $f = f'$.

Lastly, let $G\colon \Psi(X) \to C$ be a chain map. The morphisms $\Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \to \Psi(X)$ induce chain maps $G^n\colon \Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \to \Sigma^{-n} \circ \tau_0 \circ \Sigma^n(C)$, and these chain maps are compatible with the chain maps $\Sigma^{-(n+1)}\big(\mathbf{N}(\sigma_n) \circ \kappa'_{X_{n,\bullet}}\big)\colon \Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \to \Sigma^{-(n+1)}\mathbf{N}(X_{n+1,\bullet})$. By the naturality of κ' the leftmost part of the diagram

$$\begin{tikzcd}
\Sigma^{-n}\mathbf{N}(X_{n,\bullet}) \arrow[r, "\Sigma^{-(n+1)}\kappa'_{X_{n,\bullet}}"] \arrow[d, "G^n"'] & \Sigma^{-(n+1)}\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge X_{n,\bullet}\big) \arrow[r, "\Sigma^{-(n+1)}\mathbf{N}(\sigma_{n,\bullet})"] \arrow[d, "\Sigma^{-(n+1)}\mathbf{N}(\mathrm{id}_{\mathbf{F}_*(S^1_\bullet)} \wedge \Sigma^n G^n)"'] & \Sigma^{-(n+1)}\mathbf{N}(X_{n+1,\bullet}) \arrow[d, "G^{n+1}"'] \\
\Sigma^{-n}\tau_0\Sigma^n C \arrow[r, "\Sigma^{-(n+1)}\kappa'_{\mathbf{L}\tau_0\Sigma^n C}"] & \Sigma^{-(n+1)}\mathbf{N}\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L}\tau_0\Sigma^n C\big) \arrow[r, "\Sigma^{-(n+1)}\mathbf{N}((\rho'_n)_C)"] & \Sigma^{-(n+1)}\tau_0\Sigma^{n+1} C
\end{tikzcd}$$

commutes, while the rightmost part commutes by Lemma 2.4.12 in conjunction with the computations $\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L} \circ \tau_0\Sigma^n C\big)_0 = 0$ and $\big(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{L} \circ \tau_0\Sigma^n C\big)_1 \simeq \ker \partial_{-n}$. Thus the Dold–Kan correspondence gives morphisms $g_n\colon X_{n,\bullet} \to \mathbf{L}\tau_0\Sigma^n C$, and these morphisms are compatible with the structure maps. It follows that the g_ns give the desired morphism $g\colon X \to \Phi(C)$. □

Note that Ψ and Φ does not form an equivalence of categories. Although the composition $\Psi \circ \Phi$ is the identity on chain complexes, the colimit part of Ψ ensures that $\Phi \circ \Psi$ is not an isomorphism on naive modules over H$\mathbf{Z}$. To see this, consider a naive H$\mathbf{Z}$ module X where $X_{0,\bullet} = *$ and another naive H$\mathbf{Z}$ module X' such that $X_{n,\bullet} = X'_{n,\bullet}$ for $n > 0$ and $\sigma'_{0,\bullet}$ is the constant morphism to the basepoint.

Definition 2.4.14. A spectrum of C*-spaces is a functor $\mathsf{C^*\text{-}alg} \to \mathsf{Spt}$, while a morphism between spectra of C*-spaces is a natural transformation. The category of spectra of C*-spaces will be denoted by $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$.

In the *stable projective* model structure on spectra of C*-spaces, a morphism θ is a weak equivalence (resp. fibration) if θ_A is a weak equivalence (resp. fibration) in the stable model structure on Spt for all separable C*-algebras A. ♠

Observe that the free–forgetful adjunction also extends to an adjunction

$$\mathbf{F}\colon \mathsf{Spt} \rightleftarrows \text{nvmod-H}\mathbf{Z}\colon \mathbf{U}$$

which is a Quillen adjunction. Since the stable Dold–Kan correspondence is a Quillen equivalence, the proof next results follows along the same lines as the proof of Proposition 2.3.9, i.e. that the composition of left Quillen functors is again a left Quillen functor.

Proposition 2.4.15. *The stable Dold–Kan correspondence yields a Quillen adjunction*

$$(\Psi \circ \mathbf{F})_*\colon [\mathsf{C^*\text{-}alg}, \mathsf{Spt}] \rightleftarrows \mathsf{Ch}([\mathsf{C^*\text{-}alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})\colon (\mathbf{U} \circ \Phi)_*. \qquad \square$$

Similarly to the simplicial spectrum case, there is a *levelwise projective model structure* on $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$. In this model structure a morphism $\theta\colon \mathrm{X} \to \mathrm{Y}$ is a weak equivalence (resp. fibration) if the morphism $\theta_{n,\bullet}\colon \mathrm{X}_{n,\bullet} \to \mathrm{Y}_{n,\bullet}$ is a weak equivalence (resp. fibration) in the pointwise projective model structure on simplicial C*-spaces. Note that $\theta\colon \mathrm{X} \to \mathrm{Y}$ is an acyclic fibration in the levelwise projective model structure on $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ if and only if θ_A is a levelwise acyclic fibration in Spt for all separable C*-algebras A. Moreover, if $\theta\colon \mathrm{X} \to \mathrm{Y}$ is a levelwise projective weak equivalence, then θ_A is a levelwise weak equivalence for all separable C*-algebras A. So θ is also a stable projective weak equivalence.

Lemma 2.4.16. *The levelwise projective model structure has the same cofibrations as the stable projective model structure on* $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$.

Proof. Recall that $\eta\colon \mathrm{V} \to \mathrm{W}$ is a cofibration in the levelwise projective model structure if and only if η has the left lifting property with respect to all morphism θ in $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ such that θ_A is a levelwise acyclic fibrations for all C*-algebras A. Now the levelwise acyclic fibrations on Spt are precisely the acyclic fibrations of the stable model structure on Spt by [Hov01, Corollary 3.5]. So η is a levelwise projective cofibration if and only if it has the left lifting property with respect to all morphisms θ in $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ such that θ_A is a stable acyclic fibration for all separable C*-algebras A. It follows that the stable projective cofibrations coincide with the levelwise projective cofibrations. $\square$

Proposition 2.4.17. *Let* $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_l$ *and* $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_s$ *denote the category* $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ *with the levelwise projective and stable projective model structure respectively. In this case the identity functor* $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_l \to [\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_s$ *is a left Quillen functor.* $\square$

By [Hov01, Theorem 1.13] the levelwise model structure on Spt is both proper and combinatorial (recall that the Quillen model structure on $s\mathsf{Set}_*$ satisfies both of those properties). In view of [Hov01, Corollary 3.5] it follows from [Hir03, Theorem 4.1.1] that the stable model structure on Spt is left proper and combinatorial. Consequentially, by Theorem 1.4.13 both the levelwise projective model structure and the stable projective model structure on $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ are both left proper* and combinatorial. Combining this with the fact that the identity functor is a left Quillen functor from the projective levelwise model structure to the stable projective model structure hints to the fact that the stable projective model structure is a left Bousfield localisation of the levelwise projective model structure. This is indeed the case. One way to see this is to use [Dug01, Proposition 3.2]. Another method is to use a question on MathOverflow asked by Charles Rezk and answered by Denis-Charles Cisinski (`http://mathoverflow.net/questions/19313`).

Corollary 2.4.18. *There is a set $\mathscr{S}$ such that the stable projective model structure is the left Bousfield localisation of the levelwise projective model structure with respect to $\mathscr{S}$.*

Proof. For X a spectrum of C*-spaces, let $QX \to X$ be some cofibrant replacement in the levelwise projective model structure. Since the map $QX \to X$ is a levelwise projective weak equivalence, it is also a stable projective weak equivalence. So the identity functor $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_l \to [\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_s$ is homotopically surjective in the sense of [Dug01, Definition 3.1]. Since the levelwise projective model structure is proper, [Dug01, Proposition 3.2] provides a set $\mathscr{S}$ such that the left Bousfield localisation of $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_l$ with respect to $\mathscr{S}$ is $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]_s$. $\square$

In [Øst10] the stable model structure has its genesis in the Bousfield localisation of the levelwise projective model structure. So let $f\colon X \to Y$ be a morphism of C*-spectra. The morphism f is a *levelwise exact projective* weak equivalence (resp. fibration) if $f_{n,\bullet}\colon X_{n,\bullet} \to Y_{n,\bullet}$ is a weak equivalence (resp. fibration) in the exact projective model structure on C*-spaces, and similarly for *levelwise matrix invariant* and *levelwise homotopy invariant* weak equivalences (resp. fibrations). By Theorem 1.4.13 these are all left proper combinatorial model structures. The passage from the levelwise exact projective model structure to the stable exact model structure now follows from Corollary 2.4.18. Thus the *stable exact projective* (resp. *stable matrix invariant* and *stable homotopy invariant*) model structure on $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ is the left Bousfield localisation of the levelwise exact projective (resp. levelwise matrix invariant and levelwise homotopy invariant) model structure with respect to the morphisms in the set $\mathscr{S}$ of Corollary 2.4.18.

Denote by Y^Δ the "simplicial Yoneda embedding" which takes a separable C*-algebra B to the discrete simplicial C*-space given by $\mathbf{n} \mapsto \mathrm{Hom}_{\mathsf{C^*\text{-}alg}}(B, \_)$ for $\mathbf{n} \in \Delta^{\mathrm{op}}$. From the proof of [Øst10, Proposition 3.32] (modified to work with split exact sequences and pointed C*-spaces), the exact projective model structure is the left Bousfield localisation of the projective model structure with respect to the morphisms of

$$\mathscr{P} = \left\{ \operatorname{hocolim} \left(\begin{array}{c} \mathrm{Y}^\Delta(A) \to \mathrm{Y}^\Delta(A'') \\ \downarrow \\ * \end{array} \right) \to \mathrm{Y}^\Delta(A') \;\middle|\; A' \to A \to A'' \text{ is split exact} \right\}.$$

* The levelwise projective model structure is also right proper.

Let $\mathscr{P}'$ be the collection of morphism f in $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ such that $f_{n,\bullet} \in \mathscr{P}$ for all $n \geq 0$, and note that it is a set if one restricts to looking at objects A, A' and A'' from a skeleton of $\mathsf{C}^*\text{-}\mathsf{alg}$. It follows that stable exact model structure is the left Bousfield localisation of the levelwise projective model structure with respect to morphisms of $\mathscr{P}' \cup \mathscr{S}$.

The technique of Proposition 2.3.13 now gives Quillen adjunctions between the left Bousfield localised model structures on $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ and the left Bousfield localised model structures on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Proposition 2.4.19. *The stable Dold–Kan correspondence gives Quillen adjunctions between*

1. *the stable exact projective model structure on* $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ *and the split exact model structure on* $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.
2. *the stable matrix invariant model structure on* $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ *and the split exact and matrix stable model structure on* $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.
3. *the stable homotopy invariant model structure on* $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ *and the split exact, matrix stable and homotopy invariant model structure on* $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

Proof. Since the proof of the first point is virtually identical to the proofs of the second and third points, only the former will be proved. By Proposition 2.4.15 the stable Dold–Kan correspondence gives the Quillen adjunction

$$(\Psi \circ \mathbf{F})_* \colon [\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}] \rightleftarrows \mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}}) \colon (\mathbf{U} \circ \Phi)_*$$

in the unlocalised model structures. By the preceding discussion, the stable exact model structure is the left Bousfield localisation of the stable projective model structure on $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ with respect to the morphisms of $\mathscr{P}'$. Moreover, since the split exact model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$ is a left Bousfield localisation of the projective model structure, the functor $(\Psi \circ \mathbf{F})_*$ is a left Quillen functor from the stable projective model structure on $[\mathsf{C}^*\text{-}\mathsf{alg}, \mathsf{Spt}]$ to the split exact model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$.

By [Hir03, Proposition 3.3.18(1)] $(\Psi \circ \mathbf{F})_*$ is a left Quillen functor from the stable exact model structure to the split exact model structure if $(\Psi \circ \mathbf{F})_*\big(\mathrm{Q}(f)\big)$ is a split exact weak equivalence for all morphisms $f \in \mathscr{P}'$. So let $f \colon X \to Y$ be in $\mathscr{P}'$ and note that the map $\mathrm{Q}(f)$ is a weak equivalence between cofibrant objects in the levelwise exact model structure. To see this, note that the cofibrant replacement functors coincide in the levelwise exact model structure and the stable exact model structure since the latter is a left Bousfield localisation of the former. Since the diagram

$$\begin{tikzcd}
\mathrm{Q}(X) \arrow[r, two heads, "\simeq"] \arrow[d, "\mathrm{Q}(f)"] & X \arrow[d, "f"] \\
\mathrm{Q}(Y) \arrow[r, two heads, "\simeq"] & Y
\end{tikzcd}$$

commutes, $\mathrm{Q}(f)$ is a levelwise exact weak equivalence if $f \in \mathscr{P}'$.

From the above considerations and Ken Brown's lemma ([Hov99, Lemma 1.1.12]), it is enough to show that $(\Psi \circ \mathbf{F})_*$ maps acyclic cofibrations in the levelwise exact model structure to a weak equivalence in the split exact model structure. So assume $f\colon X \to Y$ is an acyclic cofibration, and note that by the left lifting property with respect to levelwise exact fibrations the morphisms $f_{0,\bullet}\colon X_{0,\bullet} \to Y_{0,\bullet}$ and $S^1_\bullet \wedge Y_{n,\bullet} \coprod_{S^1_\bullet \wedge X_{n,\bullet}} X_{n+1,\bullet} \to Y_{n+1,\bullet}$ are exact projective acyclic cofibrations* in $[\mathsf{C}^*\text{-}\mathsf{alg}, s\mathsf{Set}_*]$ for $n \geq 0$ (where $S^1_\bullet \wedge \mathrm{F}$ is the functor $A \mapsto S^1_\bullet \wedge \mathrm{F}(A)$).

In order to show that $(\Psi \circ \mathbf{F})_*(f)$ is a weak equivalence it is enough to show that there is a lift in diagrams of the form

$$\begin{tikzcd}
(\Psi \circ \mathbf{F})_*(X) \arrow[r] \arrow[d, "(\Psi\circ\mathbf{F})_*(f)"] & \mathrm{C}_1 \arrow[d, "\rho"] \\
(\Psi \circ \mathbf{F})_*(Y) \arrow[r] & \mathrm{C}_2
\end{tikzcd}$$

with $\rho\colon \mathrm{C}_1 \to \mathrm{C}_2$ a fibration in the split exact model structure on $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0}, \mathsf{Ab}]_{\mathsf{Ab}})$. Since $(\mathbf{N}\circ\mathbf{F}_*)_*$ is a left Quillen functor by Proposition 2.3.13 there is a lift in the diagram

$$\begin{tikzcd}
(\mathbf{N} \circ \mathbf{F}_*)_*(X_{0,\bullet}) \arrow[r] \arrow[d, "(\mathbf{N}\circ\mathbf{F}_*)_*(f_{0,\bullet})"] & (\Psi \circ \mathbf{F})_*(X) \arrow[r] & \mathrm{C}_1 \arrow[d, "\rho"] \\
(\mathbf{N} \circ \mathbf{F}_*)_*(Y_{0,\bullet}) \arrow[r] & (\Psi \circ \mathbf{F})_*(Y) \arrow[r] & \mathrm{C}_2.
\end{tikzcd}$$

Now assume $n > 0$ and that there is a lift λ_{n-1} in the diagram

$$\begin{tikzcd}
\Sigma^{-(n-1)}(\mathbf{N} \circ \mathbf{F}_*)_*(X_{n-1,\bullet}) \arrow[r] \arrow[d, "\Sigma^{-(n-1)}(\mathbf{N}\circ\mathbf{F}_*)_*(f_{n-1,\bullet})"'] & (\Psi \circ \mathbf{F})_*(X) \arrow[r] & \mathrm{C}_1 \arrow[d, "\rho"] \\
\Sigma^{-(n-1)}(\mathbf{N} \circ \mathbf{F}_*)_*(Y_{n-1,\bullet}) \arrow[r] & (\Psi \circ \mathbf{F})_*(Y) \arrow[r] & \mathrm{C}_2.
\end{tikzcd}$$

Denote $\mathbf{F}_*(Z_{m,\bullet})$ by Z_m and consider the diagram

$$\begin{tikzcd}
\Sigma^{-(n-1)}\mathbf{N}_*(X_{n-1}) \arrow[r] \arrow[d] & \Sigma^{-n}\mathbf{N}_*\big(\mathbf{F}_*(S^1_\bullet) \wedge X_{n-1}\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner."] & \Sigma^{-n}\mathbf{N}_*(X_n) \arrow[r] \arrow[d] & \mathrm{C}_1 \arrow[dd, "\rho"] \\
\Sigma^{-(n-1)}\mathbf{N}_*(Y_{n-1}) \arrow[r] & \Sigma^{-n}\mathbf{N}_*\big(\mathbf{F}_*(S^1_\bullet) \wedge Y_{n-1}\big) \arrow[r] & \Sigma^{-n}\mathrm{P} \arrow[d, "\eta"] & \\
& & \Sigma^{-n}\mathbf{N}_*(Y_n) \arrow[r] & \mathrm{C}_2
\end{tikzcd} \qquad (*)$$

where P is the pushout, i.e. $\mathrm{P} = \mathbf{N}_*\big(\mathbf{F}_*(S^1_\bullet) \wedge Y_{n-1}\big) \coprod_{\mathbf{N}_*\mathbf{F}_*(S^1_\bullet)\wedge X_{n-1}} \mathbf{N}_*(X_n)$. Since $(\mathbf{N} \circ \mathbf{F}_*)_*$ is a left adjoint it commutes with colimits, and by freeness there is a natural isomorphism $\mathbf{F}_{**}(S^1_\bullet \wedge Z_{n,\bullet}) \simeq \mathbf{F}_*(\mathbf{F}_*(S^1_\bullet) \wedge \mathbf{F}_*(Z_{n,\bullet}))$. Thus there are natural isomor-

* See [GJ09, Proposition X.4.13] or [Hov01, Proposition 1.14].

phisms

$$\begin{aligned}\mathrm{P} &\simeq \Big((\mathbf{N}\circ\mathbf{F}_*)_*(S^1_\bullet\wedge Y_{n,\bullet})\Big)\coprod_{(\mathbf{N}\circ\mathbf{F}_*)_*(S^1_\bullet\wedge X_{n,\bullet})}\Big((\mathbf{N}\circ\mathbf{F}_*)_*(X_{n+1,\bullet})\Big)\\ &\simeq (\mathbf{N}\circ\mathbf{F}_*)_*\Big((S^1_\bullet\wedge Y_{n,\bullet})\coprod_{S^1_\bullet\wedge X_{n,\bullet}} X_{n+1,\bullet}\Big)\end{aligned}$$

whence η is an acyclic cofibration (recall that $(\mathbf{N}\circ\mathbf{F}_*)_*$ is a left Quillen functor). Consequentially, if there is a natural transformation $\beta\colon \Sigma^{-n}\mathrm{P}\to \mathrm{C}_1$ making the diagram $(*)$ commutative then there is a lift $\lambda_n\colon \Sigma^{-n}(\mathbf{N}\circ\mathbf{F}_*)_*(X_{n,\bullet})\to\mathrm{C}_1$ compatible with λ_{n-1}.

Let $\mathrm{C}=(\mathrm{C}_n,\partial_n)=(\mathbf{N}\circ\mathbf{F}_*)_*(Y_{n-1,\bullet})$ and note that $\ker\partial_0\simeq\mathrm{C}_0$. By the (unstable) Dold–Kan correspondence and Lemma 2.4.12 the composite $\mathbf{N}((\rho'_0)_{\mathrm{C}}\circ\kappa'_{\mathbf{L}\tau_0\mathrm{C}}$ is an isomorphism, i.e. there is a section to the morphism $\Sigma^{-(n-1)}\mathbf{N}_*(Y_{n-1})\to\Sigma^{-n}\mathbf{N}_*\big(\mathbf{F}_*(S^1_\bullet)\wedge Y_{n-1}\big)$. Combining this section with λ_{n-1} and the fact that P is a pushout yields the desired morphism β.

The result now follows since $(\Psi\circ\mathbf{F})_*(Y)$ is the colimit over the morphisms $\Sigma^{-n}(\mathbf{N}\circ\mathbf{F}_*)_*(Y_{n,\bullet})\to\Sigma^{-n+1}(\mathbf{N}\circ\mathbf{F}_*)_*(Y_{n+1,\bullet})$. Thus the lifts $\lambda_n\colon\Sigma^{-n}(\mathbf{N}\circ\mathbf{F}_*)_*(Y_{n,\bullet})\to\mathrm{C}_1$ give the desired lift $(\Psi\circ\mathbf{F})_*(Y)\to\mathrm{C}_1$. Since ρ was an arbitrary fibration in the split exact model structure $(\Psi\circ\mathbf{F})_*(f)$ is an acyclic cofibration, and in particular a weak equivalence. □

As mentioned earlier, a naive module over the Eilenberg–Mac Lane spectrum of the integers is akin to a module over the ring of integers. However, a better analogue is given by a *module over the Eilenberg–Mac Lane spectrum*. This is a symmetric spectrum where the structure maps can be extended to Σ_n-equivariant simplicial group homomorphisms. The stable Dold–Kan correspondence is part of a zig-zag of Quillen equivalences between chain complexes and modules over H$\mathbf{Z}$[SS03b, Appendix B].

The zig-zag of the previous paragraph extends to a zig-zag of Quillen equivalences between $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}})$ and $[\mathsf{C}^*\text{-}\mathsf{alg},\text{mod-H}\mathbf{Z}]_{\mathsf{Ab}}$ (if the latter is equipped with the stable model structure). Moreover, one can work with symmetric spectra of C*-spaces (i.e. functors $\mathsf{C}^*\text{-}\mathsf{alg}\to\mathsf{Spt}^\Sigma$). If it is possible to reduce this zig-zag to a Quillen equivalence (with the functor $\mathsf{Ch}(\mathsf{Ab})\to\text{mod-H}\mathbf{Z}$ as the right adjoint) then this would give a Quillen adjunction between $\mathsf{Ch}([\mathsf{C}^*\text{-}\mathsf{alg}_{\mathsf{Ab}^0},\mathsf{Ab}]_{\mathsf{Ab}})$ and $[\mathsf{C}^*\text{-}\mathsf{alg},\mathsf{Spt}^\Sigma]$. However, at the present time it is unknown if this is possible.

In [Øst10, Chapter 4] the notion of a C*-spectrum is introduced. This is in some sense analogue to spectra — as spectra allow delooping of the simplicial suspension, C*-spectra allow delooping of both simplicial and C*-algebra suspensions. In order to compare C*-spectra with spectra of C*-spaces, it is necessary to introduce the smash product in $\mathsf{C}^*\text{-}\mathsf{spc}_*$. Recall that the category $s\mathsf{Set}_*$ is a closed monoidal category, so by the work of Day on internal objects, the category $\mathsf{C}^*\text{-}\mathsf{spc}_*$ is also closed monoidal. In particular there is an internal smash product* $\_\ \underline{\wedge}\ \_$ and an internal hom object $\underline{\mathrm{Hom}}$†.

* The internal tensor product is constructed in a similar way to the one in Definition 1.2.2, so $(\mathrm{F}_1\ \underline{\wedge}\ \mathrm{F}_2)(A)=\mathrm{colim}_{A_1\otimes_\sigma A_2\to A}\mathrm{F}_1(A_1)\wedge\mathrm{F}_2(A_2)$. † In $\mathsf{C}^*\text{-}\mathsf{spc}_*$ the internal hom is given by $\underline{\mathrm{Hom}}(Y,X)(A)_k=\mathrm{Nat}(Y\wedge\Delta^k_+,X(A\otimes_\sigma\_)$ where $Y\wedge\Delta^k_+$ the functor $B\mapsto Y(B)\wedge\Delta^k_+$.

Definition 2.4.20. Denote the simplicial Yoneda embedding of $\mathrm{C}_0(\mathbf{R})$ by $\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} = \mathrm{Y}^{\Delta}\big(\mathrm{C}_0(\mathbf{R})\big)$. A C*-spectrum is a collection $(X_{n,\bullet}, \sigma_{n,\bullet})_{n\geq 0}$ of pointed simplicial C*-spaces $X_{n,\bullet}$ and C*-space morphisms (called structure maps) $\sigma_{n,\bullet}\colon S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} X_{n,\bullet} \to X_{n+1,\bullet}$ (where $S^1_\bullet \wedge X_\bullet$ is the functor $A \mapsto S^1_\bullet \wedge X_\bullet(A)$). Given simplicial spectra $(X_{n,\bullet}, \sigma_{n,\bullet})$ and $(X'_{n,\bullet}, \sigma'_{n,\bullet})$, a *morphism of C*-spectra* $(f_{n,\bullet})\colon (X_{n,\bullet}, \sigma_{n,\bullet}) \to (X'_{n,\bullet}, \sigma'_{n,\bullet})$ is a collection $(f_{n,\bullet})_{n\geq 0}$ of morphisms $f_{n,\bullet}\colon X_{n,\bullet} \to X'_{n,\bullet}$ in $\mathsf{C^*\text{-}spc}_*$ such that the diagram

$$\begin{tikzcd}
S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} X_{n,\bullet} \arrow[r, "\sigma_{n,\bullet}"] \arrow[d, "\mathrm{id}_{S^1_\bullet} \mathbin{\underline{\triangle}} \mathrm{id}_{\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}} \mathbin{\underline{\triangle}} f_{n,\bullet}"'] & X_{n+1,\bullet} \arrow[d, "f_{n+1,\bullet}"] \\
S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} X'_{n,\bullet} \arrow[r, "\sigma'_{n,\bullet}"] & X'_{n+1,\bullet}
\end{tikzcd}$$

commutes for all $n \geq 0$. The category of C*-spectra is denoted by Spt_C. ♠

The functor $\mathrm{C}\colon [\mathsf{C^*\text{-}alg}, \mathsf{Spt}] \to \mathsf{Spt}_C$ takes a spectrum of C*-spaces $(X_{n,\bullet}, \sigma_{n,\bullet})_{n\geq 0}$ to the C*-spectrum whose nth space is $X_{n,\bullet} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}$ and where the structure maps are $(\sigma_{n,\bullet}) \mathbin{\underline{\triangle}} \mu_n^{-1}) \circ (\mathrm{id}_{S^1_\bullet} \wedge \tau)$ with τ the twist map $\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} X_{n,\bullet} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)} \to X_{n,\bullet} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}$ and μ the isomorphism $\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^{n+1})} \to \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}$ obtained from the morphism $\nu\colon \mathrm{C}_0(\mathbf{R}^n) \otimes_\sigma \mathrm{C}_0(\mathbf{R}) \to \mathrm{C}_0(\mathbf{R}^{n+1})$ given by $\nu(f \otimes g)(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n) g(x_{n+1})$.

Conversely, the functor $\mathrm{V}\colon \mathsf{Spt}_C \to [\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$ takes a C*-spectrum $(X'_{n,\bullet}, \sigma'_{n,\bullet})_{n\geq 0}$ to the spectrum of C*-spaces whose nth space is $\underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}, X_{n,\bullet})$ while the structure maps $S^1_\bullet \wedge \underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}, X_{n,\bullet}) \to \underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^{n+1})}, X_{n+1,\bullet})$ are the result of hom–smash adjunction applied to the composite

$$\begin{tikzcd}
S^1_\bullet \wedge \underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}, X_{n,\bullet}) \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^{n+1})} \arrow[d, "\mathrm{id}_{S^1_\bullet} \wedge \mathrm{id}_{\underline{\mathrm{Hom}}} \mathbin{\underline{\triangle}} \mu"'] & \\
S^1_\bullet \wedge \underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}, X_{n,\bullet}) \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)} \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \arrow[d, "\mathrm{id}_{S^1_\bullet} \wedge \tau^{-1}"'] & \\
S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} \underline{\mathrm{Hom}}(\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}, X_{n,\bullet}) \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)} \arrow[r, "\mathrm{id}_{S^1_\bullet} \wedge \mathrm{id}_{\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}} \mathbin{\underline{\triangle}} \mathrm{ev}_{\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}}"] & S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})} \mathbin{\underline{\triangle}} X_{n,\bullet} \arrow[d, "\sigma'_{n,\bullet}"'] \\
& X_{n+1,\bullet}
\end{tikzcd}$$

(where the map $\mathrm{ev}_{\mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R}^n)}}$ is the counit of the $\underline{\mathrm{Hom}}$–$\underline{\triangle}$ adjunction of pointed C*-spaces).

A tedious computation using the $\underline{\mathrm{Hom}}$–$\underline{\triangle}$ adjunction for pointed C*-spaces now shows that there is an natural bijection $\mathrm{Hom}_{\mathsf{Spt}_C}\big(\mathrm{C}(X), Z\big) \to \mathrm{Hom}_{[\mathsf{C^*\text{-}alg},\mathsf{Spt}]}\big(X, \mathrm{V}(Z)\big)$, i.e. C is the left adjoint of V.

The stable model structure on C*-spectra is tailored so that $\_ \mathbin{\underline{\triangle}} S^1_\bullet \wedge \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}$ is invertible in the homotopy category. It follows that both $\_ \wedge S^1_\bullet$ and $\_ \mathbin{\underline{\triangle}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}$ are invertible in the homotopy category. On the other hand, the stable projective model structure on spectra of C*-space is rigged so that $\_\wedge S^1_\bullet$ is invertible in the homotopy category. Thus, in order to get closer to C*-spectra the operation $\_ \mathbin{\underline{\wedge}} \mathrm{Y}^{\Delta}_{\mathrm{C}_0(\mathbf{R})}$ has to be be made invertible in $[\mathsf{C^*\text{-}alg}, \mathsf{Spt}]$.

2.5 A slice filtration for KK

In [Øst10, Chapter 6] Østvær discusses the slice filtration for the stable C*-homotopy category. This is a filtration whose genesis lies in motivic stable homotopy theory and the work of Voevodsky[Voe02]. The aim of this section is to study a similar filtration related to KK using the techniques of [HK06, Section 1].

The relationship between pointed simplicial C*-spaces and pointed simplicial C*-spaces with KK-transfers is discussed in [Øst10, Section 5.2]. Due to the Dold–Kan correspondence and the fact that KK is additive, one could equally well speak about the category of connected chain complexes of additive functors from KK to Ab. Note that the results in category theory of part I also works in the additive case (with the obvious modifications).

Definition 2.5.1. Given an additive category A, the category of additive functors from A to Ab will be denoted by $[\mathsf{A}, \mathsf{Ab}]_{ad}$. ♠

Consider the object $\mathrm{Hom}_{\mathsf{KK}}(\mathbf{C}, \_)^0 = \mathrm{KK}(\mathbf{C}, \_)^0$ in $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$, i.e. the object which is $\mathrm{KK}(\mathbf{C}, \_)$ in degree 0 and otherwise 0. For any $\mathrm{F} \in \mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$ and separable C*-algebra A the internal tensor product of $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$ satisfies

$$\begin{aligned}\big(\mathrm{F}\underline{\otimes}\mathrm{Hom}_{\mathsf{KK}}(\mathbf{C}, \_)^0\big)(A)_n &\simeq \operatorname*{colim}_{A_1\otimes_\sigma A_2\to A} \mathrm{F}(A_1)_n \otimes \mathrm{KK}(\mathbf{C}, A_2)\\ &\simeq \operatorname*{colim}_{B\otimes_\sigma \mathbf{C}\to A} \mathrm{F}(B)_n \otimes \mathrm{KK}(\mathbf{C}, \mathbf{C})\\ &\simeq \mathrm{F}(A)_n\end{aligned}$$

since $\mathrm{KK}(\mathbf{C}, \_)^0$ is concentrated in degree 0, $\mathrm{KK}(\mathbf{C}, \mathbf{C}) \simeq \mathbf{Z}$ and $\mathbf{C}$ is the unit for the spatial tensor product in $\mathsf{C^*\text{-}alg}$. On a related note,

$$\begin{aligned}\underline{\mathrm{Hom}}\big(\mathrm{Hom}_{\mathsf{KK}}(\mathbf{C}, \_)^0, \mathrm{F}\big)(A)_n &\simeq \mathrm{Nat}\left(\mathrm{O}\circ \mathrm{Hom}_{\mathsf{KK}}(\mathbf{C}, \_)^0, \mathrm{O}\circ \Sigma^{-n}\mathrm{F}(A\otimes_\sigma \_)\right)\\ &\simeq \mathrm{Nat}\left(\mathrm{KK}(\mathbf{C}, \_), \mathrm{F}(A\otimes_\sigma \_)_n\right)\\ &\simeq \mathrm{F}(A)_n.\end{aligned}$$

Recall that the suspension functor for C*-algebras are given by $\mathrm{C}_0(\mathbf{R})\otimes_\sigma\_$ while the suspension functor for chain complexes, Σ, can be realised at $\mathbf{S}^1(\mathbf{Z}) \otimes_T \_$ where $\mathbf{S}^1(\mathbf{Z})$ is the chain complex which in degree 1 is $\mathbf{Z}$, and elsewhere 0. The combination of these suspensions forms the basis for the Tate object.

Definition 2.5.2. The *Tate object* of $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$ is $\mathrm{Z}(1) = \Sigma^1\mathrm{Hom}_{\mathsf{KK}}(\mathrm{C}_0(\mathbf{R}), \_)^0$ i.e. $\mathrm{Z}(1)$ is $\mathrm{Hom}_{\mathsf{KK}}(\mathrm{C}_0(\mathbf{R}), \_)$ in degree 1 and 0 otherwise.

Let $\mathrm{Z} = \mathrm{Z}(0) = \mathrm{Hom}_{\mathsf{KK}}(\mathbf{C}, \_)^0$ and $\mathrm{Z}(n) = \mathrm{Z}(1)^{\underline{\otimes} n}$ for $n > 2$. Since $\underline{\otimes}$ extends $\otimes_\sigma$, it follows that $\mathrm{Z}(n)$ is $\mathrm{Hom}_{\mathsf{KK}}(\mathrm{C}_0(\mathbf{R}^n), \_)$ in degree n and 0 otherwise. For $n \geq 0$ and F an object of $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$, the nth *Tate twist* of F is given by $\mathrm{F}(n) = \mathrm{F}\underline{\otimes}\mathrm{Z}(n)$. ♠

With the Tate twist at hand, one can for each $n \in \mathbf{N}$ form the full subcategory $\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})(n)$ of $\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})$ generated by objects on the form $\mathrm{F}(n)$ for some $\mathrm{F} \in \mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})$.

Definition 2.5.3. The functor $\nu^{\geq n}\colon \mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad}) \to \mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})(n)$ is defined by $\nu^{\geq n} = \underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)(n)$ for any $n \geq 0$. ♠

Notice that an application of the hom–tensor adjunction to the identity morphism of $\underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)$ yields a natural transformation $a^n\colon \underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)(n) \to \mathrm{id}_{\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})}$. Re-applying the adjunction to $a^n\colon \underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)\underline{\otimes}\mathrm{Z}(1)\underline{\otimes}\mathrm{Z}(n-1) \to \mathrm{id}_{\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})}$ gives a natural transformation $\tilde{f}^{n-1}\colon \underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)(1) \to \underline{\mathrm{Hom}}\big(\mathrm{Z}(n-1), \_\big)$, and taking the Day convolution of $\tilde{f}^{n-1}$ with the identity of $\mathrm{Z}(n-1)$ gives the natural transformation $f^{n-1}\colon \nu^{\geq n} \to \nu^{\geq n-1}$, i.e. $f^{n-1} = \tilde{f}^{n-1}\underline{\otimes}\mathrm{id}_{\mathrm{Z}(n-1)}$. The relation $a^{n-1} \circ f^{n-1} = a^n$ now follows from a tedious computation*.

Note that a^n is the counit of the hom–tensor adjunction involving $\mathrm{Z}(n)$. Thus the isomorphism $\mathrm{Hom}_{\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})}\big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{Z}(n), \mathrm{G})\big) \to \mathrm{Hom}_{\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})}\big(\mathrm{F}(n), \mathrm{G}\big)$ is given by $\zeta \mapsto a^n_{\mathrm{G}} \circ (\zeta\underline{\otimes}\mathrm{id}_{\mathrm{Z}(n)})$. Let $b^n\colon \mathrm{id}_{\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})} \to \underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_(n)\big)$ denote the unit of the hom-tensor adjunction, i.e. b^n_{F} is the result using the adjunction on the morphism $\mathrm{id}_{\mathrm{F}(n)}$.

Lemma 2.5.4. *The morphism $b^1_{\mathrm{G}}\colon \mathrm{G} \to \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{G}(1)\big)$ is an isomorphism, so the functor $\_(1)\colon \mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad}) \to \mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})$ is fully faithful.*

Proof. Let F and G be objects of $\mathsf{Ch}([\mathsf{KK},\mathsf{Ab}]_{ad})$, and note that by the internal hom–tensor adjunction

$$\mathrm{Nat}\Big(\mathrm{F}(1), \mathrm{G}(1)\Big) \simeq \mathrm{Nat}\Big(\mathrm{F}, \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{G}(1)\big)\Big).$$

Thus the second statement of the lemma follows directly from the first statement.

By Proposition 1.2.13, if A is a separable C*-algebra, $i \in \mathbf{Z}$ and $x \in \mathrm{G}(A)_i$ then

$$(b^1_{\mathrm{G}})^i_A(x) \in \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{G}(1)\big)(A)_i = \mathrm{Nat}\Big(\mathrm{O}\circ\mathrm{Z}(1)(\_), \mathrm{O}\circ\Sigma^{-i}\circ\big(\mathrm{G}\underline{\otimes}\mathrm{Z}(1)\big)(A\otimes_\sigma\_)\Big)$$

is the natural transformation given by $(b^1_{\mathrm{G}})^i_A(x)^j_B(y) = x \otimes y \in \mathrm{G}(A)_i \otimes \mathrm{Z}(1)(B)_j$ for B a separable C*-algebra, $j \in \mathbf{Z}$, $y \in \mathrm{Z}(1)(B)_j$ and where $\mathrm{G}(A)_i \otimes \mathrm{Z}(1)(B)_j$ is viewed as a summand of $\big(\mathrm{G}(A) \otimes_T \mathrm{Z}(1)(B)\big)_{i+j}$. Moreover, since $\mathrm{Z}(1)_j = 0$ for $j \neq 1$ and

* The index n of f^n is different from the one used in [HK06, p. 2]. This is done so that it coincide with the index used in [HK06, p. 3].

$\mathrm{Z}(1)_1 = \mathrm{Hom}_{\mathsf{KK}}(\mathrm{C}_0(\mathbf{R}), \_)$ by definition,

$$\begin{aligned}
\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{G}(1)\big)(A)_i &\simeq \mathrm{Nat}\Big(\mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}), _\big), \big(\mathrm{G}\underline{\otimes}\mathrm{Z}(1)\big)(A\otimes_\sigma_)\Big)_{i+1} \\
&\simeq \big(\mathrm{G}\underline{\otimes}\mathrm{Z}(1)\big)\big(A\otimes_\sigma \mathrm{C}_0(\mathbf{R})\big)_{i+1} \\
&\simeq \operatorname*{colim}_{A_1\otimes_\sigma A_2 \to A\otimes_\sigma \mathrm{C}_0(\mathbf{R})} \big(\mathrm{G}(A_1)\otimes_T \mathrm{Z}(1)(A_2)\big)_{1+i} \\
&\simeq \operatorname*{colim}_{A_1\otimes_\sigma A_2 \to A\otimes_\sigma \mathrm{C}_0(\mathbf{R})} \mathrm{G}(A_1)_i \otimes \mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}), A_2\big) \\
&\simeq \operatorname*{colim}_{B\otimes_\sigma \mathrm{C}_0(\mathbf{R}) \to A\otimes_\sigma \mathrm{C}_0(\mathbf{R})} \mathrm{G}(B)_i \otimes \mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}), \mathrm{C}_0(\mathbf{R})\big) \\
&\simeq \operatorname*{colim}_{B\otimes_\sigma \mathrm{C}_0(\mathbf{R}) \to A\otimes_\sigma \mathrm{C}_0(\mathbf{R})} \mathrm{G}(B)_i \\
&\simeq \mathrm{G}(A)_i
\end{aligned}$$

where the last isomorphism comes from the invertibility of suspensions in KK. Under the above isomorphisms $(b^1_{\mathrm{G}})^i_A(x)$ corresponds to the element $x \in \mathrm{G}(A)_i$ since the isomorphism $\mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}), \mathrm{C}_0(\mathbf{R})\big) \simeq \mathbf{Z}$ sends $\mathrm{id}_{\mathrm{C}_0(\mathbf{R})}$ to 1. $\square$

A central point in the construction of a slice filtration is distinguished triangles. For $[\mathsf{KK}, \mathsf{Ab}]_{ad}$ the triangulated category in question is the derived category $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$. Notice that for an object F of $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$, a separable C*-algebra A and an integer k the relations

$$\underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \mathrm{F}\big)(A)_k \simeq \mathrm{Nat}\big(\mathrm{Hom}_{\mathsf{KK}}(\mathrm{C}_0(\mathbf{R}^n), \_), \mathrm{F}(A\otimes_\sigma\_)_{n-k}\big) \simeq \mathrm{F}\big(A\otimes_\sigma \mathrm{C}_0(\mathbf{R}^n)\big)_{n-k}$$

and

$$\big(\mathrm{F}\underline{\otimes}\mathrm{Z}(n)\big)(A)_k \simeq \operatorname*{colim}_{A_1\otimes_\sigma A_2\to A} \mathrm{F}(A_1)_{k-n} \otimes \mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}^n), A_2\big)$$

hold. Consequentially, both $\underline{\mathrm{Hom}}\big(\mathrm{Z}(n), \_\big)$ and $\_\underline{\otimes}\mathrm{Z}(n)$ preserves quasi-isomorphisms, so they induce functors $D([\mathsf{KK}, \mathsf{Ab}]_{ad}) \to D([\mathsf{KK}, \mathsf{Ab}]_{ad})$. Moreover, the above relations also shows that the induced functors are triangulated. In particular this implies that the functors $\_(n)$ and $\nu^{\geq n}$ induce triangulated functors $D([\mathsf{KK}, \mathsf{Ab}]_{ad}) \to D([\mathsf{KK}, \mathsf{Ab}]_{ad})(n)$. A last piece of needed knowledge is that the $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ is closed symmetric monoidal by Theorem 1.4.16. It follows that the hom–tensor adjunction of $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$ gives a derived hom–tensor adjunction in $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$.

Proposition 2.5.5. *The induced functor* $\_(1)\colon D([\mathsf{KK}, \mathsf{Ab}]_{ad}) \to D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ *is fully faithful.*

Proof. Recall that a morphism F to G in $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ is an equivalence class of diagrams $\mathrm{F} \xleftarrow{\sim} \mathrm{K} \longrightarrow \mathrm{G}$ in $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$. Moreover, if $\mathrm{F}(1) \xleftarrow{\sim} \mathrm{K} \longrightarrow \mathrm{G}(1)$ is a morphism of $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ an application of $\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \_\big)$ give the top row of

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{F}(1)\big) \arrow[d, "\simeq"] & \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{K}\big) \arrow[l, "\sim"'] \arrow[r] \arrow[d, equal] & \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{G}(1)\big) \arrow[d, "\simeq"] \\
\mathrm{F} & \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{K}\big) \arrow[l, "\sim"'] \arrow[r] & \mathrm{G}
\end{tikzcd}$$

while the vertical isomorphisms are consequences of the proof of Lemma 2.5.4. Applying the functor _(1) to the lower row of the above diagram gives the morphism $\mathrm{F}(1) \xleftarrow{\sim} \underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{K}\big)(1) \longrightarrow \mathrm{G}(1)$. Thus it remains to show that given a morphism $\zeta \colon \mathrm{K} \to \mathrm{F}(1)$ then the diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{K}\big)(1) \arrow[rr, "a^1_{\mathrm{K}}"] \arrow[d, "\underline{\mathrm{Hom}}(\mathrm{Z}(1), _)(\zeta) \underline{\otimes} \mathrm{id}_{\mathrm{Z}(1)}"'] & & \mathrm{K} \arrow[d, "\zeta"] \\
\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{F}(1)\big)(1) \arrow[rr, "(b^1_{\mathrm{F}})^{-1} \underline{\otimes} \mathrm{id}_{\mathrm{Z}(1)}"] & & \mathrm{F}(1)
\end{tikzcd}$$

commutes. By the naturality of $a^1_{\_}$ and the invertibility of b^1_{F} it is enough to show that the composition $\big(b^1_{\mathrm{F}} \underline{\otimes} \mathrm{Z}(1)\big) \circ a^1_{\mathrm{F}}$ is the identity on $\underline{\mathrm{Hom}}\big(\mathrm{Z}(1), \mathrm{F}(1)\big)(1)$. This readily follows from the definitions of a^1 and b^1.

The same technique shows that if two morphisms $\mathrm{F} \xleftarrow{\sim} \mathrm{K}_1 \longrightarrow \mathrm{G}$ and $\mathrm{F} \xleftarrow{\sim} \mathrm{K}_2 \longrightarrow \mathrm{G}$ become the same morphism in $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ after an application of _(1), i.e. there is a commutative diagram

$$\begin{tikzcd}
& \mathrm{K}_1(1) \arrow[dl, "\sim"'] \arrow[dr] & \\
\mathrm{F}(1) & \mathrm{K} \arrow[l, "\sim"', "q"] \arrow[u] \arrow[r] \arrow[d] & \mathrm{G}(1) \\
& \mathrm{K}_2(1) \arrow[ul, "\sim"] \arrow[ur] &
\end{tikzcd}$$

in $\mathsf{Ch}([\mathsf{KK}, \mathsf{Ab}]_{ad})$, then they were the same morphism before the application of _(1). $\square$

Corollary 2.5.6 ([HK06, Proposition 1.1])**.** *The functor $\nu^{\geq n}$ is right adjoint to the inclusion $D([\mathsf{KK}, \mathsf{Ab}]_{ad})(n) \hookrightarrow D([\mathsf{KK}, \mathsf{Ab}]_{ad})$.*

Proof. For objects F and G of $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ Proposition 2.5.5 and the hom–tensor adjunction yields

$$\mathrm{Hom}_D\big(\mathrm{F}(n), \nu^{\geq n}\mathrm{G}\big) \simeq \mathrm{Hom}_D\big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{Z}(n), \mathrm{G})\big) \simeq \mathrm{Hom}_D\big(\mathrm{F}(n), \mathrm{G}\big). \qquad \square$$

Recall that a^n is the counit of the hom–tensor adjunction involving $\mathrm{Z}(n)$ so the isomorphism $\mathrm{Hom}_D\big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{Z}(n), \mathrm{G})\big) \to \mathrm{Hom}_D\big(\mathrm{F}(n), \mathrm{G}\big)$ is given by $\zeta \mapsto a^n \circ (\zeta \otimes \mathrm{id}_{\mathrm{Z}(n)})$. On the other hand, the isomorphism $\mathrm{Hom}_D\big(\mathrm{F}, \underline{\mathrm{Hom}}(\mathrm{Z}(n), \mathrm{G})\big) \to \mathrm{Hom}_D\big(\mathrm{F}(n), \nu^{\geq n}\mathrm{G}\big)$ is given by $\zeta \mapsto \zeta \otimes \mathrm{id}_{\mathrm{Z}(n)}$. It follows that the isomorphism $\mathrm{Hom}_D\big(\mathrm{F}(n), \nu^{\geq n}\mathrm{G}\big) \to \mathrm{Hom}_D\big(\mathrm{F}(n), \mathrm{G}\big)$ is given by $\zeta \mapsto a^n \circ \zeta$.

Definition 2.5.7. The subcategory $\nu_{<n}D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ of $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ is the full subcategory whose objects F are such that $\nu^{\geq n}\mathrm{F} \simeq 0$. ♠

In order to proceed with the construction of a filtration, there is a need for several adjoint functors. In [BBD82] the concept of a *t-structure* was introduced, and this is precisely the tool needed. An English version of the theory can be found in [GM03, Section IV.4].

Definition 2.5.8. A *t-structure* on a triangulated category T is a pair of full subcategories $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ satisfying the following "orthogonality" conditions:

1. If T is an object of $\mathsf{T}^{\leq 0}$ then so is ΣT, and if ΣT is an object of $\mathsf{T}^{\geq 0}$ then so is T.
2. If X is an object of $\mathsf{T}^{\leq 0}$ and Y is an object of $\mathsf{T}^{\geq 0}$ then $\mathrm{Hom}_{\mathsf{T}}(X, \Sigma^{-1}Y) = 0$.
3. For any object T of T there exists a distinguished triangle $X \to T \to \Sigma^{-1}Y \to \Sigma X$ with X an object of $\mathsf{T}^{\leq 0}$ and Y an object of $\mathsf{T}^{\geq 0}$. ♠

As mentioned before, the raison d'être for t-structures is the creation of adjunctions, and the next proposition makes this manifest.

Proposition 2.5.9 ([GM03, Lemma IV.4.5])**.** *Given a t-structure* $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ *on a triangulated category* T *then*

- *there exists a functor* $\tau_{\leq 0} \colon \mathsf{T} \to \mathsf{T}^{\leq 0}$ *(resp. a functor* $\tau_{\geq 0} \colon \mathsf{T} \to \mathsf{T}^{\geq 0}$*) which is right (resp. left) adjoint to the corresponding inclusion.*
- *for any object* T *of* T *there exists a distinguished triangle*

$$\tau_{\leq 0}T \to T \to \Sigma^{-1}\tau_{\geq 0}\Sigma T \to \Sigma\tau_{\leq 0}T$$

where $\tau_{\leq 0}T \to T$ *is the counit in the adjunction involving* $\tau_{\leq 0}$ *and* $T \to \Sigma^{-1}\tau_{\geq 0}\Sigma T$ *is the (chain complex) desuspension of the unit in the adjunction involving* $\tau_{\geq 0}$*. Moreover any two distinguished triangles* $X \to T \to \Sigma^{-1}Y \to \Sigma X$ *with* X *an object of* $\mathsf{T}^{\leq 0}$ *and* Y *an object of* $\mathsf{T}^{\geq 0}$ *are canonically isomorphic.*

Proof. This is proved in [GM03, Lemma IV.4.5], but their second bullet point is slightly different. They work with the full subcategory $\mathsf{T}^{\geq 1}$, where $Z \in \mathsf{T}^{\geq 1}$ if and only if $\Sigma Z \in \mathsf{T}^{\geq 0}$, and a corresponding left adjoint $\tau_{\geq 1} \colon \mathsf{T} \to \mathsf{T}^{\geq 1}$ instead of $\Sigma^{-1}\tau_{\geq 0}\Sigma$. However, since $\mathrm{Hom}_{\mathsf{T}^{\geq 1}}(Z', Z'') \simeq \mathrm{Hom}_{\mathsf{T}^{\geq 0}}(\Sigma Z', \Sigma Z'')$ it follows that $\mathrm{Hom}_{\mathsf{T}^{\geq 1}}(\tau_{\geq 1}X, Z) \simeq \mathrm{Hom}_{\mathsf{T}}(X, Z) \simeq \mathrm{Hom}_{\mathsf{T}}(\Sigma X, \Sigma Z) \simeq \mathrm{Hom}_{\mathsf{T}^{\geq 0}}(\tau_{\geq 0}\Sigma X, \Sigma Z) \simeq \mathrm{Hom}_{\mathsf{T}^{\geq 1}}(\Sigma^{-1}\tau_{\geq 0}\Sigma X, Z)$, so by the uniqueness of left adjoint functors $\tau_{\geq 1} \simeq \Sigma^{-1}\tau_{\geq 0}\Sigma$.

The statement regarding the counit and unit follows by inspecting the proof of [GM03, Lemma IV.4.5]. □

Corollary 2.5.10 ([HK06, Corollary 1.4 (i) and (ii)])**.** *There is a t-structure on the category* $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ *given by the pair* $\big(D([\mathsf{KK}, \mathsf{Ab}]_{ad})(n), \nu_{<n}D([\mathsf{KK}, \mathsf{Ab}]_{ad})\big)$*. Thus the inclusion* $\nu_{<n}D([\mathsf{KK}, \mathsf{Ab}]_{ad}) \hookrightarrow D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ *has a left adjoint denoted by* $\nu_{<n}$*, and for any object* F *of* $D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ *there is a distinguished triangle*

$$\nu^{\geq n}\mathrm{F} \xrightarrow{a^n_{\mathrm{F}}} \mathrm{F} \xrightarrow{(a_n)_{\mathrm{F}}} \nu_{<n}\mathrm{F} \longrightarrow \Sigma\nu^{\geq n}\mathrm{F}.$$

Proof. First observe that $\Sigma\nu^{\geq n}\Sigma^{-1} \simeq \nu^{\geq n}$ so $\nu_{<n} \simeq \Sigma^{-1}\nu_{<n}\Sigma$. Moreover, both of the subcategories $D([\mathsf{KK}, \mathsf{Ab}]_{ad})(n)$ and $\nu_{<n}D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ are (chain complex) suspension stable. Thus it remains to check two things in order to have a t-structure. The first is that if $\mathrm{F}(n) \in D([\mathsf{KK}, \mathsf{Ab}]_{ad})(n)$ and $\mathrm{G} \in \nu_{<n}D([\mathsf{KK}, \mathsf{Ab}]_{ad})$ then $\mathrm{Hom}_D(\mathrm{F}(1), \Sigma^{-1}\mathrm{G}) \simeq 0$. This is a direct consequence of Corollary 2.5.6. The second thing to check is that for

any F in $D([\mathsf{KK},\mathsf{Ab}]_{ad})$ there is a distinguished triangle $\mathrm{F}' \to \mathrm{F} \to \mathrm{F}'' \to \Sigma\mathrm{F}$ with $\mathrm{F}' \in D([\mathsf{KK},\mathsf{Ab}]_{ad})(n)$ and $\mathrm{F}'' \in \nu_{<n}D([\mathsf{KK},\mathsf{Ab}]_{ad})\big)$. To see that this is the case observe that the triangle

$$\nu^{\geq n}\mathrm{F} \xrightarrow{a^n_{\mathrm{F}}} \mathrm{F} \longrightarrow \mathrm{cone}\,(a^n_{\mathrm{F}}) \longrightarrow \Sigma\nu^{\geq n}\mathrm{F}$$

is distinguished. So since $\nu^{\geq n}$ is a triangulated functor it is enough to show that $\nu^{\geq n}a^n$ is an isomorphism. By Proposition 2.5.5 this reduces to showing that

$$\underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\_\big)(a^n)\colon \underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\nu^{\geq n}\_\big) \to \underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\_\big)$$

is an isomorphism. If A is a separable C*-algebra and $k \in \mathbf{Z}$, then

$$\begin{aligned}
\underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\nu^{\geq n}\mathrm{F}\big)(A)_k &= \mathrm{Nat}\Big(\mathrm{O}\circ\mathrm{Z}(n)(_),\big(\mathrm{O}\circ\Sigma^{-k}\nu^{\geq n}\mathrm{F}\big)(A\otimes_\sigma_)\Big)\\
&\simeq \mathrm{Nat}\Big(\,\mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}^n),_\big),\nu^{\geq n}\mathrm{F}(A\otimes_\sigma_)_{n+k}\Big)\\
&\simeq \Big(\underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\mathrm{F}\big)\underline{\otimes}\mathrm{Z}(n)\Big)\big(A\otimes_\sigma\mathrm{C}_0(\mathbf{R}^n)\big)_{n+k}\\
&\simeq \underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\mathrm{F}\big)(A)_k\otimes\mathrm{Hom}_{\mathsf{KK}}\big(\mathrm{C}_0(\mathbf{R}^n),\mathrm{C}_0(\mathbf{R}^n)\big)\\
&\simeq \underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\mathrm{F}\big)(A)_k
\end{aligned}$$

and this isomorphism is given by $\underline{\mathrm{Hom}}\big(\mathrm{Z}(n),\_\big)(a^n)$. □

Now, the morphisms $f^n_{\mathrm{F}}\colon \nu^{\geq n+1}\mathrm{F} \to \nu^{\geq n}\mathrm{F}$ and $\mathrm{id}_{\mathrm{F}}\colon \mathrm{F}\to\mathrm{F}$ give rise to a morphism $(f'_n)_{\mathrm{F}}\colon \mathrm{cone}\,(a^{n+1}_{\mathrm{F}}) \to \mathrm{cone}\,(a^n_{\mathrm{F}})$. Combining this with the canonical isomorphism $(f''_n)_{\mathrm{F}}\colon \nu_{<n}\mathrm{F} \to \mathrm{cone}\,(a^n_{\mathrm{F}})$ yields the morphism

$$(f_n)_{\mathrm{F}} = (f''_n)^{-1}_{\mathrm{F}}\circ(f'_n)_{\mathrm{F}}\circ(f''_{n+1})_{\mathrm{F}}\colon \nu_{<n+1}\mathrm{F} \to \nu_{<n}\mathrm{F}.$$

This is natural in F, so there is a morphism $f_n\colon \nu_{<n+1}\to\nu_{<n}$. The identity $f_n\circ a_{n+1} = a_n$ now follows from the commutative diagram

$$\begin{tikzcd}
\nu^{\geq n+1}\mathrm{F} \arrow[r, "a^{n+1}_{\mathrm{F}}"] \arrow[d, "f^n_{\mathrm{F}}"] & \mathrm{F} \arrow[r, "(a_{n+1})_{\mathrm{F}}"] \arrow[d, equal, "\mathrm{id}_{\mathrm{F}}"] & \nu_{<n+1}\mathrm{F} \arrow[r] \arrow[d, "(f_n)_{\mathrm{F}}"] & \Sigma\nu^{\geq n+1}\mathrm{F} \arrow[d, "\Sigma f^n_{\mathrm{F}}"] \\
\nu^{\geq n}\mathrm{F} \arrow[r, "a^n_{\mathrm{F}}"] & \mathrm{F} \arrow[r, "(a_n)_{\mathrm{F}}"] & \nu_{<n}\mathrm{F} \arrow[r] & \Sigma\nu^{\geq n}\mathrm{F}.
\end{tikzcd}$$

By Axiom T3 for a triangulated category (the octahedron axiom) applied to braid of distinguished triangles generated by $a^{n+1}_{\mathrm{F}} = a^n_{\mathrm{F}}\circ f^n_{\mathrm{F}}$ there is an isomorphism $\mathrm{cone}\,(f^n_{\mathrm{F}}) \simeq \Sigma^{-1}\mathrm{cone}\,((f_n)_{\mathrm{F}})$. Let $\nu_n\mathrm{F} = \Sigma^{-1}\mathrm{cone}\,((f_n)_{\mathrm{F}})$, $\nu_{\leq n} = \nu_{<n+1}$ and consider the distinguished triangle

$$\nu_n\mathrm{F} \longrightarrow \nu_{\leq n}\mathrm{F} \xrightarrow{(f_n)_{\mathrm{F}}} \nu_{\leq n-1}\mathrm{F} \longrightarrow \mathrm{cone}\,((f_n)_{\mathrm{F}}).$$

For an object G of $D([\mathsf{KK},\mathsf{Ab}]_{ad})$ this gives the long-exact sequence

$$\begin{tikzcd}
\mathrm{Hom}_D(\nu_n\mathrm{F},\Sigma^{n+m-1}\mathrm{G}) \arrow[d] & & \\
\mathrm{Hom}_D(\nu_{\leq n-1}\mathrm{F},\Sigma^{n+m}\mathrm{G}) \arrow[r, "(f_n)^*_{\mathrm{F}}"] & \mathrm{Hom}_D(\nu_{\leq n}\mathrm{F},\Sigma^{n+m}\mathrm{G}) \arrow[r] & \mathrm{Hom}_D(\nu_n\mathrm{F},\Sigma^{n+m}\mathrm{G}) \arrow[d] \\
& & \mathrm{Hom}_D(\nu_{\leq n-1}\mathrm{F},\Sigma^{n+m+1}\mathrm{G})
\end{tikzcd}$$

i.e. part of an unrolled exact couple. Moreover, the isomorphism $\nu^{\geq 0}\mathrm{F} \simeq \mathrm{F}$ implies that $\nu_{<0}\mathrm{F} = \nu_{\leq -1}\mathrm{F}$ is zero in $D([\mathsf{KK},\mathsf{Ab}]_{ad})$. Define $A^{s,t} = \mathrm{Hom}_D(\nu_{\leq -s}\mathrm{F},\Sigma^{t-s}\mathrm{G})$ and $E^{s,t} = \mathrm{Hom}_D(\nu_{-s}\mathrm{F},\Sigma^{t-s}\mathrm{G})$ for $s \leq 0$ and note that by letting $A^{s,t} = E^{s,t} = 0$ for $s > 0$ the above long exact sequence is extended to an unrolled exact couple.

However, due to Bott periodicity the resulting spectral sequence is somewhat uninteresting. To see this, note that

$$\begin{aligned}
(\nu^{\geq 2n}\mathrm{F})(A)_k &\simeq \operatorname*{colim}_{A_1\otimes_\sigma A_2\to A} \mathrm{Nat}\left(\mathrm{Hom}_{\mathsf{KK}}\left(\mathrm{C}_0(\mathbf{R}^{2n}),_\right),\mathrm{F}(A_1\otimes_\sigma _)_k\right)\otimes \mathrm{Hom}_{\mathsf{KK}}\left(\mathrm{C}_0(\mathbf{R}^{2n}),A_2\right)\\
&\simeq \operatorname*{colim}_{B\otimes_\sigma \mathrm{C}_0(\mathbf{R}^{2n})\to A} \mathrm{F}\big(B\otimes_\sigma \mathrm{C}_0(\mathbf{R}^{2n})\big) \simeq \mathrm{F}(A)_k
\end{aligned}$$

where the last isomorphism follows from the Bott periodicity isomorphism. Consequentially $\nu_{\leq 2n-1}\mathrm{F} \simeq 0$ so $A^{2s-1,t} \simeq 0$ for all s.

A Smallness of model categories

In order to prove that something is a model category, there is a device known as "Quillen's small object argument" that can be used to prove the factorisation property (Axiom MC4). This device requires a notion of "smallness", whose definition is a mixture of categories and ordinals.

Definition A.1. Let C be a category, I a collection of arrows in C and λ an ordinal. A λ-*sequence* in C is a diagram

$$C_0 \to C_1 \to C_2 \to \cdots \to C_\alpha \to C_{\alpha+1} \to \cdots \tag{$*$}$$

(with $\alpha + 1 < \lambda$) in C such that for every limit ordinal* $\gamma < \lambda$, the induced morphism $\operatorname{colim}_{\alpha<\gamma} C_\alpha \to C_\gamma$ is an isomorphism. A λ-*sequence in* I is a λ-sequence where all the arrows in the sequence $(*)$ are from I.

The *composition* of a λ-sequence is the morphism $C_0 \to \operatorname{colim}_{\alpha<\lambda} C_\alpha$ (it is assumed that all the above mentioned colimits exist in C). A *transfinite composition* of morphisms in I is the composition of a λ-sequence in I. ♠

Example A.2. In Set, the category of sets, the transfinite composition of any collection of injective maps is again injective by transfinite induction. Assume

$$C_0 \to C_1 \to C_2 \to \cdots \to C_\alpha \to C_{\alpha+1} \to \cdots$$

is a λ-sequence of injective maps. If $\lambda = 0$ or λ is a successor ordinal, then the composition is clearly injective.

So assume λ is a limit ordinal and let $f_0^\alpha \colon C_0 \to C_\alpha$ be the composition of the α-sequence starting at C_0. By assumption f_0^α is injective for all $\alpha < \lambda$. Now if $f_0^\lambda(x) = f_0^\lambda(y)$, that is $\langle x, C_0\rangle = \langle y, C_0\rangle$ in $\operatorname{colim}_{\beta<\lambda} C_\beta$, there is some $\alpha < \lambda$ such that $f_0^\alpha(x) = f_0^\alpha(y)$. Thus $x = y$. ♣

Definition A.3. Let κ be a cardinal number. An object C of C is κ-*small relative to* I if for every regular cardinal† $\lambda \geq \kappa$ and every λ-sequence in I, the set-map $\operatorname{colim}_{\alpha<\lambda} \operatorname{Hom}_{\mathsf{C}}(C, C_\alpha) \to \operatorname{Hom}_{\mathsf{C}}(C, \operatorname{colim}_{\alpha<\lambda} C_\alpha)$ is an isomorphism. If there is a cardinal number κ such that C is κ-small relative to I, then C is *small relative to* I. ♠

Assume C is κ-small relative to I. Clearly, if $\kappa' > \kappa$, C is also κ'-small relative to I. On the other hand, if $\kappa' < \kappa$, C does not need to be κ'-small relative to I.

* A limit ordinal is an ordinal that is neither 0 nor the successor of another ordinal. For instance, any infinite cardinal is a limit ordinal. † A cardinal λ is regular if the cofinality of λ is λ. Thus λ is regular if and only if for all ordinals $\gamma \subset \lambda$ such that γ has cardinality less than λ, one has $\sup \gamma < \lambda$.

Example A.4. Consider Top, the category of topological spaces, and let I be all inclusions in Top. Consider a λ-sequence

$$C_0 \to C_1 \to C_2 \to \cdots \to C_\alpha \to C_{\alpha+1} \to \cdots$$

in I (where λ is a regular cardinal to be specified later), and let $f\colon C \to \operatorname{colim}_{\alpha<\lambda} C_\alpha$ be continuous.

If C is compact, then C is $\aleph_0$-small relative to I. To see this, assume $\lambda \geq \aleph_0$, and let $U_1, \ldots, U_n$ be an open cover* of $f(C)$. For each U_i there is a corresponding C_{α_i} such that $C_{\alpha_i} \cap U_i = U_i$. Let $\alpha = \sup_{i=1,\ldots,n} \alpha_i$. Then $\alpha < \lambda$ and f factors through C_α. This gives the surjectivity of the map $\operatorname{colim}_{\alpha<\lambda} \operatorname{Hom}_{\mathsf{C}}(C, C_\alpha) \to \operatorname{Hom}_{\mathsf{C}}(C, \operatorname{colim}_{\alpha<\lambda} C_\alpha)$, and injectivity follows by a similar argument.

If C is Lindelöf (i.e. any open cover has a countable refinement, see [Mun99, p. 192]) then C is $\aleph_1$-small† relative to I. To see this, assume $\lambda \geq \aleph_1$, and let $\{U_n\}_{n\in\mathbf{N}}$ be a countable open cover* of $f(C)$. For each U_n there is a corresponding C_{α_n} such that $C_{\alpha_n} \cap U_n = U_n$. Let $\alpha = \sup_{n\in\mathbf{N}} \alpha_n$ and note that f factors through C_α. The claim now follows since $\lambda > \aleph_0$ is regular, whence $\alpha < \lambda$.

Note that $\mathbf{N}$ with the discrete topology is Lindelöf but not compact. This space is *not* $\aleph_0$-small relative to I. To see this let $C_n = \{0, \ldots, n\}$ with the discrete topology and $C_n \to C_{n+1}$ be the obvious inclusion. Then $\mathbf{N} = \operatorname{colim}_{n\in\mathbf{N}} C_n$, but the identity on $\mathbf{N}$, $\mathrm{id}_{\mathbf{N}}$, does not factor through any C_n. ♣

Definition A.5. A morphism in C is in *I-cell* or is a *relative I-cell complex* if it is the transfinite composition of pushout of elements in I. Thus if $f\colon A \to B$ is in I-cell, then there is an ordinal λ and morphisms $f_\alpha\colon C_\alpha \to C'_\alpha$ in I for $\alpha + 1 < \lambda$ such that

$$\begin{tikzcd} C_\alpha \arrow[r] \arrow[d, "f_\alpha"'] \arrow[dr, phantom, "\ulcorner"] & A_\alpha \arrow[d] \\ C'_\alpha \arrow[r] & A_{\alpha+1} \end{tikzcd}$$

is a pushout, and $A = A_0 \to A_1 \to A_2 \to \cdots$ is a λ-sequence with f as composition.

The collection I *permits the small object argument* if the domains of elements of I are small relative to I-cell. ♠

If $f\colon A \to B$ is in I-cell then f is in I-cof. To see this, suppose $g\colon A' \to B'$ has the right lifting property with respect to I and $h\colon A \to A'$, $k\colon B \to B'$ are morphisms making the evident diagram commute. By induction on λ there is a lifting in the diagram, since for $\lambda = \alpha + 1$ a successor ordinal B is a pushout of f_α, while for λ a limit ordinal B is the colimit over $\alpha < \lambda$.

The converse does not always hold, but the proof of the small object argument shows that any map f can be factorised as $p \circ g$ with g in I-cell and p in I-inj. If f is in I-cof, this implies that f is a retract of g.

---

* The open cover should consist of elements from the subbasis for the colimit topology. † By [Kun06, Lemma 10.37] $\aleph_1 = \aleph_0^+$, the least cardinal number greater that $\aleph_0$, is regular (assuming the Axiom of choice).

Note that the cofibrations are closed under retractions and composition by definition. Since the cofibrations are precisely the morphisms having the left lifting property with respect to all acyclic fibrations it follows that cofibrations are closed under pushouts and transfinite composition.

Definition A.6. A model category C is *cofibrantly generated* if there exist *sets* of cofibrations I and acyclic cofibrations J such that the fibrations are precisely the J-injectives, the acyclic fibrations are precisely the I-injectives, and the domains in I and J are small relative to I-cell and J-cell respectively. ♠

Note that in a cofibrantly generated model category the class of cofibrations coincides with I-cof and the class of acyclic cofibrations coincides with J-cof. Moreover, any cofibration is a retract of an I-cell complex and any acyclic cofibration is a retract of a J-cell complex. In order to determine if a category has a cofibrantly generated model structure, the following result by Kan is helpful:

Theorem A.7. *[Hov99, Theorem 2.1.19] Suppose C is a bicomplete category and that W is a class of morphisms in C that is saturated* and closed under retracts. Let I and J be sets of morphisms of C such that the following conditions hold.*

- *The domains of I are small relative to I-cell.*
- *The domains of J are small relative to J-cell.*
- *The class J-cell is contained in both W and I-cof.*
- *The class I-inj is contained in both W and J-inj.*
- *Either the intersection of W and I-cof is contained in J-cof or the intersection of W and J-inj is contained in I-inj.*

Then there is a cofibrantly generated model structure on C with weak equivalences W, generating cofibrations I, and generating acyclic cofibrations J.

Another smallness criterion on a category is that of being locally presentable:

Definition A.8. Let C be a category and λ a regular cardinal.

A poset P is called λ-*directed* if every subset of P with cardinality less than λ has an upper bound. View P as a category in the standard way†. A λ-*directed colimit* in C is the colimit of a functor from P to C.

An object C of C is λ-*presentable* if the functor $\mathrm{Hom}_{\mathsf{C}}(C, \_)$ preserves λ-directed colimits (i.e. it takes λ-directed colimits to λ-directed colimits and it also preserves the limiting cones of functors from λ-directed posets. For the definition of a limiting cone see [Mac98, p. 67]).

The category C is *locally λ-presentable* if it is cocomplete and has a set Λ of λ-presentable objects such that every object of C is the λ-directed colimit of objects from Λ. If C is locally λ-presentable for some λ, then C is *locally presentable.* ♠

* A class of morphism W is saturated if it satisfies the “two of three property”, that is if $f, g, g \circ f$ are morphisms, and any two of them is in W, then so is the third. † I.e. there is a unique morphism from p_1 to p_2 if $p_1 \leq p_2$, and no morphism from p_1 to p_0 if $p_0 < p_1$.

It can be hard to show that a category is locally presentable. However, if one accepts *Vopěnka's principle* things are much easier:

Definition A.9. *Vopěnka's principle* says that if C is a locally presentable category and S is a full subcategory of C with only identity morphisms, then the objects of S form a set. ♠

The above definition is only one of several equivalent definitions. Several alternative formulations can be found in [AR94, Remark 6.2 (1)]. Vopěnka's principle is a large cardinal axiom in the sense that it implies the existence of an extendible cardinal [Jec78, Lemma 33.15]. On the other hand, if there exist a huge cardinal then Vopěnka's principle is consistent with ZFC [Jec78, Lemma 33.16].

Proposition A.10 ([AR94, Corollary 6.37])**.** *Vopěnka's principle implies that a cocomplete category* C *is locally presentable if and only if there is a* small *subcategory* S *of* C *such that any object in* C *is the colimit of a diagram in* S*.*

Definition A.11. A model category is *combinatorial* if it is both *cofibrantly generated* and *locally presentable.* ♠

Bibliography

[AH59] M. F. Atiyah and F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. *Bull. Amer. Math. Soc.*, 65:276–281, 1959.

[AH61] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, RI, 1961.

[AR94] J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*, volume 189 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1994.

[Ati70] M. F. Atiyah. Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo, 1970.

[Bar10] C. Barwick. On left and right model categories and left and right Bousfield localizations. *Homology, Homotopy Appl.*, 12(2):245–320, 2010.

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[BDF73] L. G. Brown, R. G. Douglas, and P. A. Fillmore. Unitary equivalence modulo the compact operators and extensions of C^*-algebras. In *Proceedings of a Conference on Operator Theory (Dalhousie Univ., Halifax, N.S., 1973)*, volume 345 of *Lecture Notes in Mathematics*, pages 58–128, Berlin, Heidelberg, New York, 1973. Springer-Verlag.

[BDF77] L. G. Brown, R. G. Douglas, and P. A. Fillmore. Extensions of C^*-algebras and K-homology. *Ann. of Math. (2)*, 105(2):265–324, 1977.

[BF78] A. K. Bousfield and E. M. Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II*, volume 658 of *Lecture Notes in Mathematics*, pages 80–130. Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[Bla98] B. Blackadar. *K-Theory for Operator Algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.

[Bor94] F. Borceux. *Handbook of categorical algebra 2: Categories and structures*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994.

[Bou75] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975.

[Bro73] K. S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Trans. Amer. Math. Soc.*, 186:419–458, 1973.

[BS58] A. Borel and J.-P. Serre. Le théorème de Riemann-Roch. *Bull. Soc. Math. France*, 86:97–136, 1958.

[CD09] D.-C. Cisinski and F. Déglise. Local and stable homological algebra in Grothendieck abelian categories. *Homology, Homotopy Appl.*, 11(1):219–260, 2009.

[CH90] A. Connes and N. Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):101–106, 1990.

[CH02] J. D. Christensen and M. Hovey. Quillen model structures for relative homological algebra. *Math. Proc. Cambridge Philos. Soc.*, 133(2):261–293, 2002.

[CMR07] J. Cuntz, R. Meyer, and J. M. Rosenberg. *Topological and Bivariant K-Theory*, volume 36 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2007.

[Cun84] J. Cuntz. K-theory and C^*-algebras. In *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, volume 1046 of *Lecture Notes in Mathematics*, pages 55–79. Springer-Verlag, Berlin, Heidelberg, New York, 1984.

[Cun87] J. Cuntz. A new look at KK-theory. *K-Theory*, 1(1):31–51, 1987.

[Cun97] J. Cuntz. Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter. *Doc. Math.*, 2:213–261, 1997.

[Cun98] J. Cuntz. A General Construction of Bivariant K-theories on the Category of C*-algebras. *Contemp. Math.*, 228:31–43, 1998.

[Day70] B. J. Day. On Closed Categories of Functors. In *Reports of the Midwest Category Seminar IV*, volume 137 of *Lecture Notes in Mathematics*, pages 1–38. Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[Dol58] A. Dold. Homology of symmetric products and other functors of complexes. *Ann. of Math. (2)*, 68:54–80, 1958.

[DS95] W. G. Dwyer and J. Spalinski. Homotopy theories and model categories. In *Handbook of Algebraic Topology*, pages 73–126. North Holland, 1995.

[Dug01] D. Dugger. Combinatorial model categories have presentations. *Adv. Math.*, 164(1):177–201, 2001.

[Fre03] P. Freyd. Abelian categories. *Repr. Theory Appl. Categ.*, (3):1–190, 2003.

[Gel41] I. M. Gelfand. Normierte Ringe. *Rec. Math. [Mat. Sbornik] N. S.*, 9 (51):3–24, 1941.

[GJ09] P. G. Goerss and J. F. Jardine. *Simplicial Homotopy Theory.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, Boston, Berlin, 2009.

[GM03] S. I. Gelfand and Y. I. Manin. *Methods of homological algebra.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, Heidelberg, second edition, 2003.

[Hat10] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 13th printing edition, 2010.

[Hig87] N. Higson. A characterization of KK-theory. *Pacific J. Math.*, 126(2):253–276, 1987.

[Hig90] N. Higson. Categories of fractions and excision in KK-theory. *J. Pure Appl. Algebra*, 65(2):119–138, 1990.

[Hir03] P. S. Hirschhorn. *Model Categories and Their Localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[HK06] A. Huber and B. Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006.

[Hov99] M. Hovey. *Model Categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 1999.

[Hov01] M. Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[Jar87] J. F. Jardine. Simplicial presheaves. *J. Pure Appl. Algebra*, 47(1):35–87, 1987.

[Jec78] T. Jech. *Set Theory.* Pure and Applied Mathematics. Academic Press, New York, NY, 1978.

[Kas80] G. G. Kasparov. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(3):571–636, 719, 1980. English translation: *Math. USSR Izv.* 16(3):513–572, 1981.

[Kas95] G. G. Kasparov. K-theory, group C^*-algebras, and higher signatures (conspectus). In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Mathematical Society Lecture Note Series*, pages 101–146. Cambridge University Press, Cambridge, 1995. (First distributed in 1981).

[Kun06] K. Kunen. *Set Theory: An Introduction To Independence Proofs*, volume 102 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 2006. Reprint of 1980 edition from North Holland.

[Lur09] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mac95] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, Heidelberg and New York, reprint of 1975 edition, 1995.

[Mac98] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, NY, second edition, 1998.

[Mar83] H. R. Margolis. *Spectra and the Steenrod Algebra*, volume 29 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1983.

[May01] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[MMSS01] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley. Model categories of diagram spectra. *Proc. London Math. Soc. (3)*, 82(2):441–512, 2001.

[MN06] R. Meyer and R. Nest. The Baum-Connes conjecture via localisation of categories. *Topology*, 45(2):209–259, 2006.

[Moo55] J. C. Moore. Homotopie des complexes monoïdaux, i. In *Algèbre d'Eilenberg-Maclane et homotopie*, volume 7 no. 2 of *Séminaire Henri Cartan*. Secrétariat mathématique, 1954–1955. Exposé No. 18, 8 p.

[Mun99] J. R. Munkres. *Topology*. Prentice Hall, Upper Saddle River, NJ, second edition, 1999.

[MV99] F. Morel and V. Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Nee91] A. Neeman. Some new axioms for triangulated categories. *J. Algebra*, 139(1):221–255, 1991.

[Nee01] A. Neeman. *Triangulated Categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001.

[Øst10] P. A. Østvær. *Homotopy Theory of C^*-Algebras*. Frontiers in Mathematics. Birkhäuser Verlag/Springer Basel, Basel, 2010.

[Pup58] D. Puppe. Homotopiemengen und ihre induzierten Abbildungen. I. *Math. Z.*, 69:299–344, 1958.

[Pup62] D. Puppe. On the formal structure of stable homotopy theory. In *Colloquium on Algebraic Topology, August 1-10 1962, Aarhus Universitet*, pages 65–71. Matematisk Institut, Aarhus Universitet, 1962.

[Qui67] D. G. Quillen. *Homotopical algebra*, volume 43 of *Lecture Notes in Mathematics*. Springer, Berlin, Heidelberg, New York, 1967.

[Rob87] A. Robinson. The extraordinary derived category. *Math. Z.*, 196(2):231–238, 1987.

[Sch84] C. Schochet. Topological methods for C^*-algebras. III. Axiomatic homology. *Pacific J. Math.*, 114(2):399–445, 1984.

[Shi07] B. Shipley. $H\mathbb{Z}$-algebra spectra are differential graded algebras. *Amer. J. Math.*, 129(2):351–379, 2007.

[SS00] S. Schwede and B. Shipley. Algebras and modules in monoidal model categories. *Proc. London Math. Soc. (3)*, 80(2):491–511, 2000.

[SS03a] S. Schwede and B. Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol.*, 3:287–334, 2003.

[SS03b] S. Schwede and B. Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003.

[Str72] A. Strøm. The homotopy category is a homotopy category. *Arch. Math. (Basel)*, 23:435–441, 1972.

[SW53] E. H. Spanier and J. H. C. Whitehead. A first approximation to homotopy theory. *Proc. Nat. Acad. Sci. U. S. A.*, 39:655–660, 1953.

[Swa62] R. G. Swan. Vector bundles and projective modules. *Trans. Amer. Math. Soc.*, 105:264–277, 1962.

[Tay75] J. L. Taylor. Banach algebras and topology. In *Algebras in analysis (Proc. Instructional Conf. and NATO Advanced Study Inst., Birmingham, 1973)*, pages 118–186. Academic Press, London, 1975.

[Tho98] K. Thomsen. The universal property of equivariant KK-theory. *J. Reine Angew. Math.*, 504:55–71, 1998.

[Uuy11] O. Uuye. Homological Algebra for C*-algebras, 2011, arXiv:1011.2926v3 [math.KT]. To appear in J. Noncommut. Geom.

[Ver77] J.-L. Verdier. Catégories dérivées (état 0). In *Séminaire de Géométrie Algébrique du Bois Marie SGA $4\frac{1}{2}$: Cohomologie Etale*, volume 569 of *Lecture Notes in Mathematics*, pages 262–311, Berlin, Heidelberg, New York, 1977. Springer-Verlag.

[Voe02] V. Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press, Somerville, MA, 2002.

[Weg93] N. E. Wegge-Olsen. *K-Theory and C*-Algebras: A Friendly Approach.* Oxford Science Publications. Oxford University Press, Oxford, 1993.

[Wei95] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge studies in advanced mathematics.* Cambridge University Press, Cambridge, 1995.

[Whi62] G. W. Whitehead. Generalized homology theories. *Trans. Amer. Math. Soc.*, 102:227–283, 1962.

[Woo66] R. Wood. Banach algebras and Bott periodicity. *Topology*, 4:371–389, 1965/1966.